Anderson transitions in Euclidean random matrix models

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PROJET FINANCÉ PAR L'ANR - PROJECT FUNDED BY THE ANR
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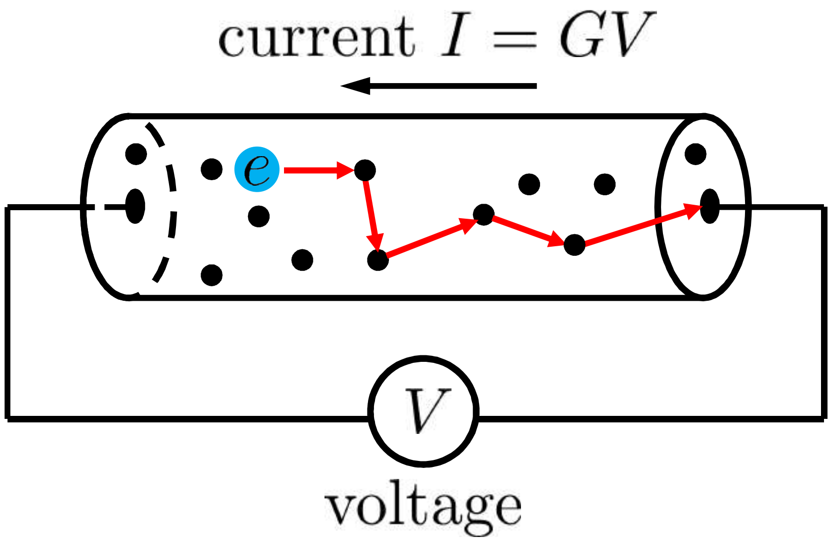
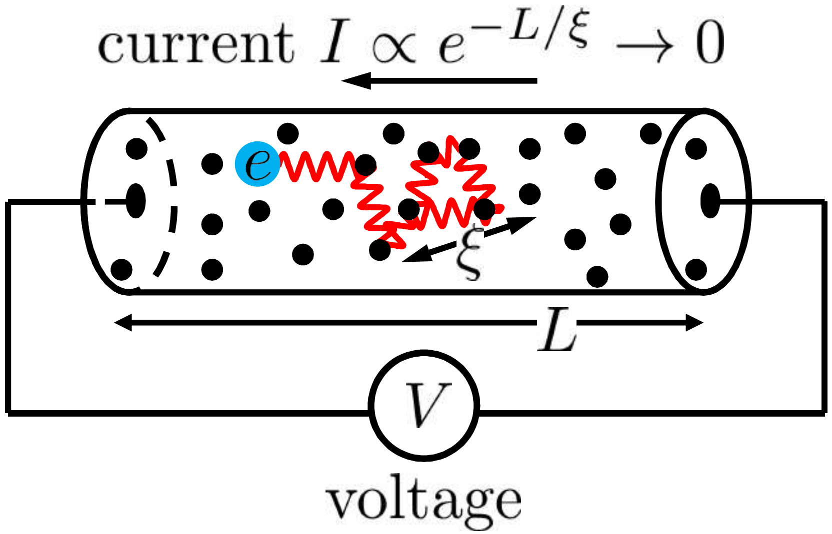
Anderson localization

Metal



Anderson insulator



Conductance $G = 1/R$

High T

Few impurities

Metal-insulator transition in 3D

$T \to 0$

Add impurities

$G \propto e^{-L/\xi} \to 0$

ξ — localization length

Anderson, PR **109**, 1492 (1958)

Absence of transport versus localization

Green's function

$$\left\{\nabla^2 + \frac{2m}{\hbar^2}\left[E - V(\mathbf{r})\right]\right\} G(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')$$

Eigenmodes

$$\left\{\nabla^2 + \frac{2m}{\hbar^2}\left[E_n - V(\mathbf{r})\right]\right\} \psi_n(\mathbf{r}) = 0$$

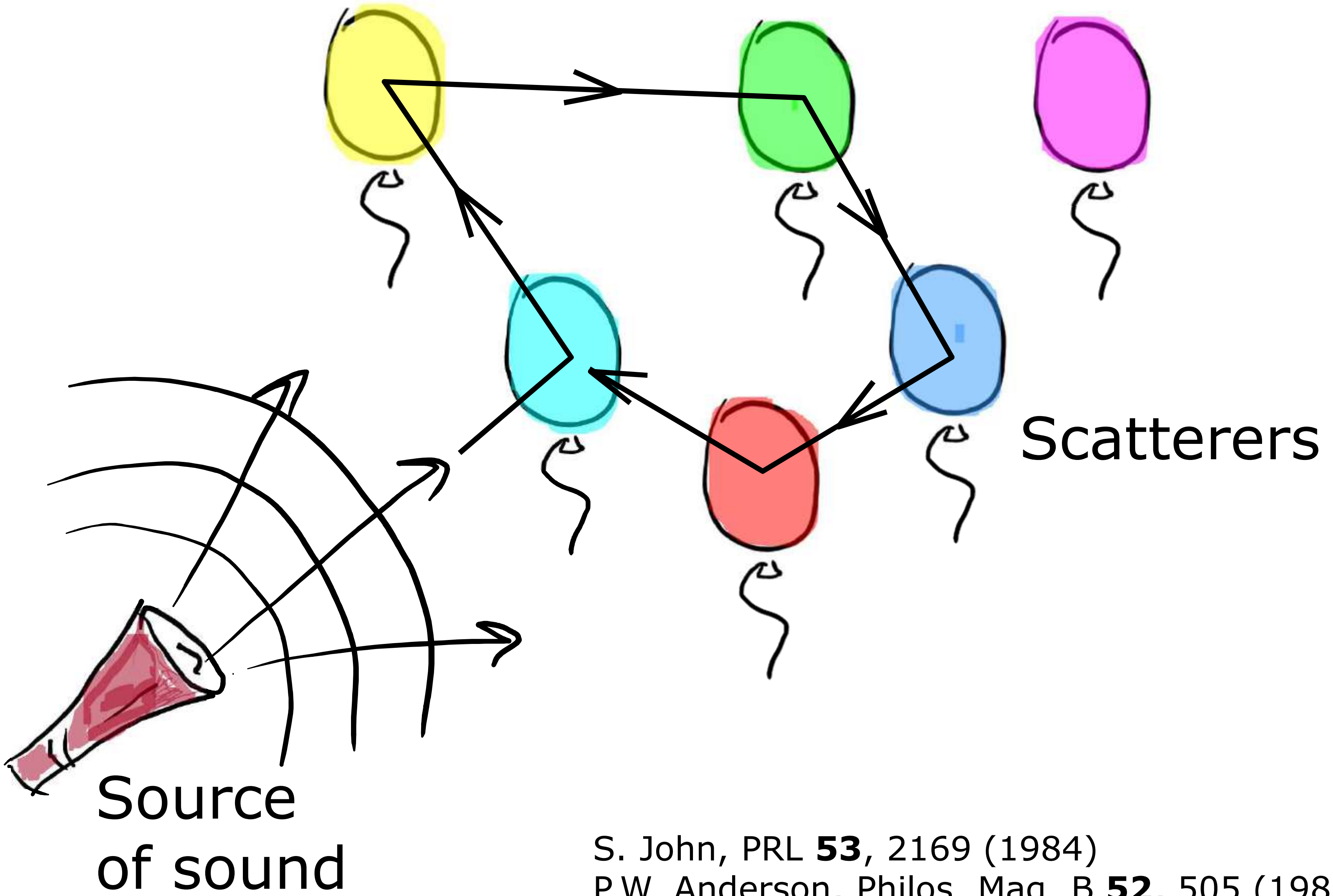
$$G(\mathbf{r},\mathbf{r}') = \frac{\hbar^2}{2m}\sum_n \frac{\psi_n(\mathbf{r})\psi_n^*(\mathbf{r}')}{E - E_n}$$

Transport $\Leftarrow$ Localization

Localized modes do not contribute to transport

No diffusion $\overset{?}{\Rightarrow}$ Localization

Can we do the same with "classical" waves?



S. John, PRL **53**, 2169 (1984)
P.W. Anderson, Philos. Mag. B **52**, 505 (1985)

Yes we can! ...in 1D and 2D

Quasi-1D: Chabanov et al., Nature **404**, 850 (2000)

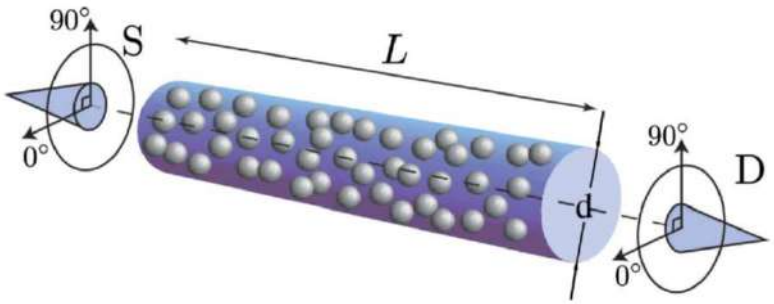
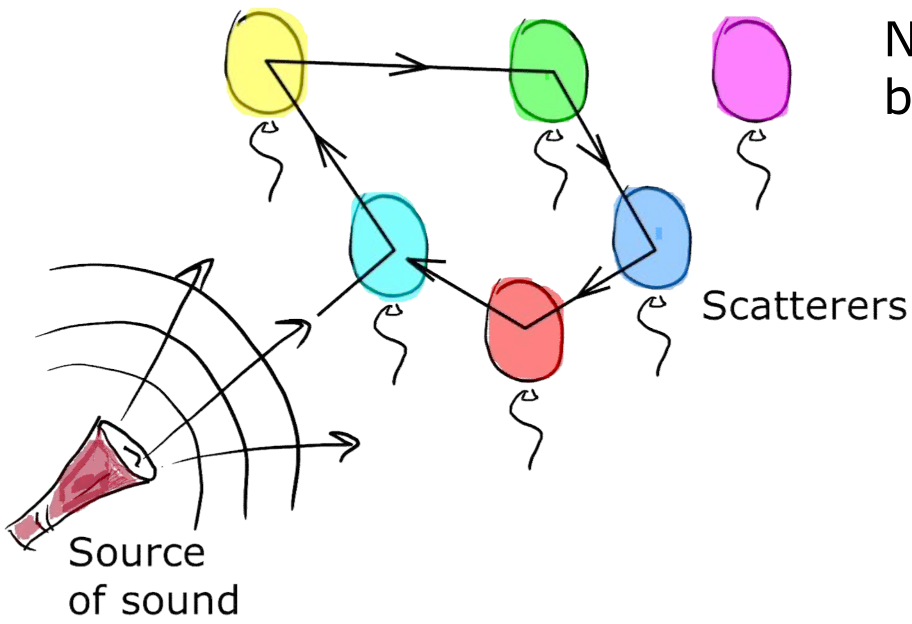


Figure from Cherroret et al., PRB **80**, 045118 (2009)

2D: Weaver, Wave Motion **12**, 129 (1990)
Dalichaouch et al., Nature **354**, 53 (1991)
Schwartz et al., Nature **446**, 52 (2007)
…

Yes we can! A real challenge in 3D

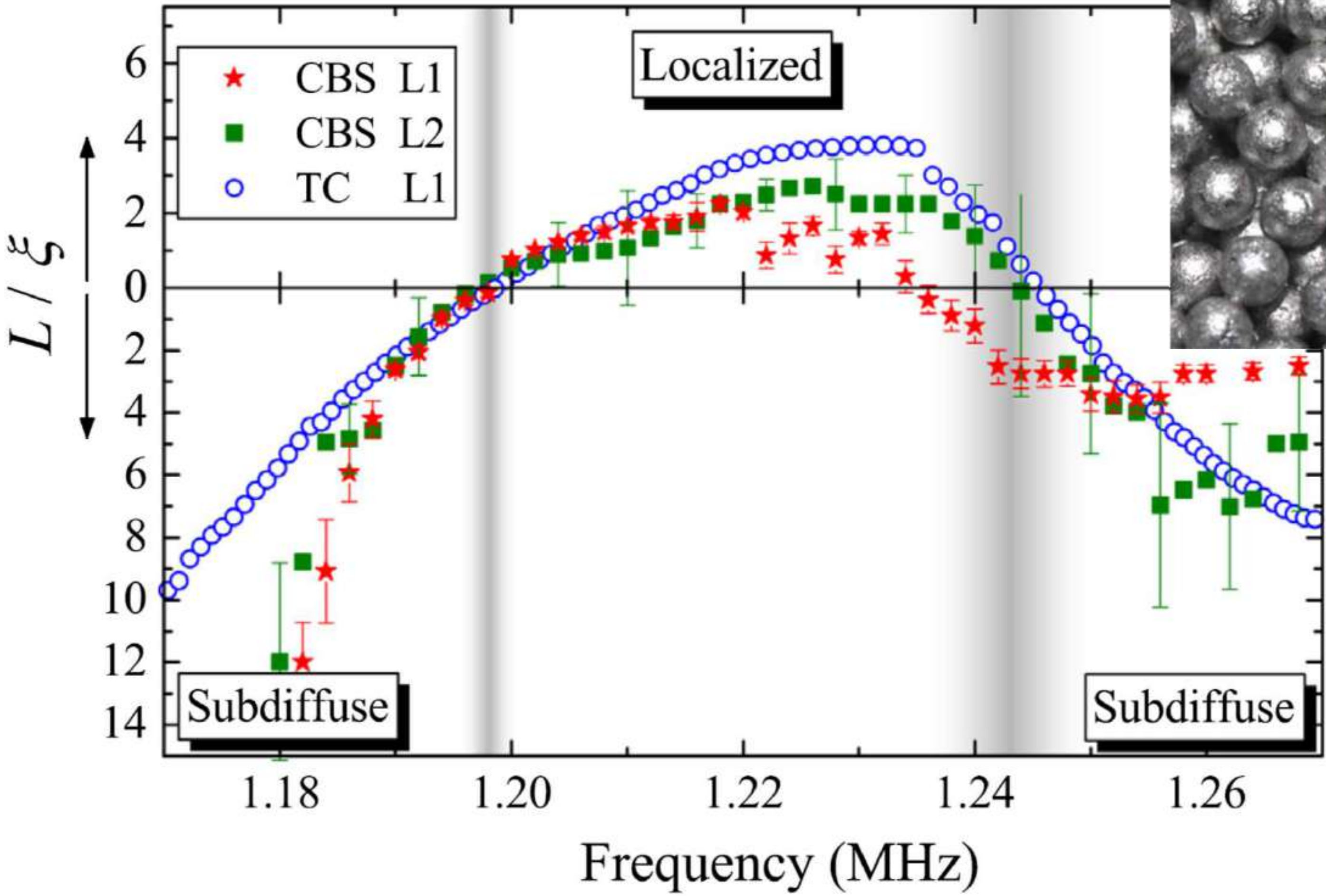


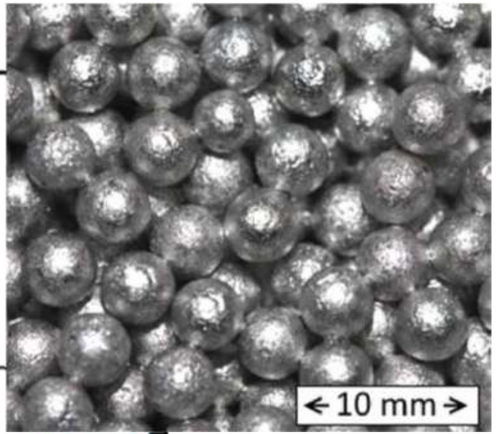
No Anderson localization of sound by balloons…

… but sound can be localized by **stronger** scattering objects (aluminum beads) that are **densely packed**

Hu et al., Nature Physics **4**, 945 (2008)

Anderson localization of elastic waves in 3D





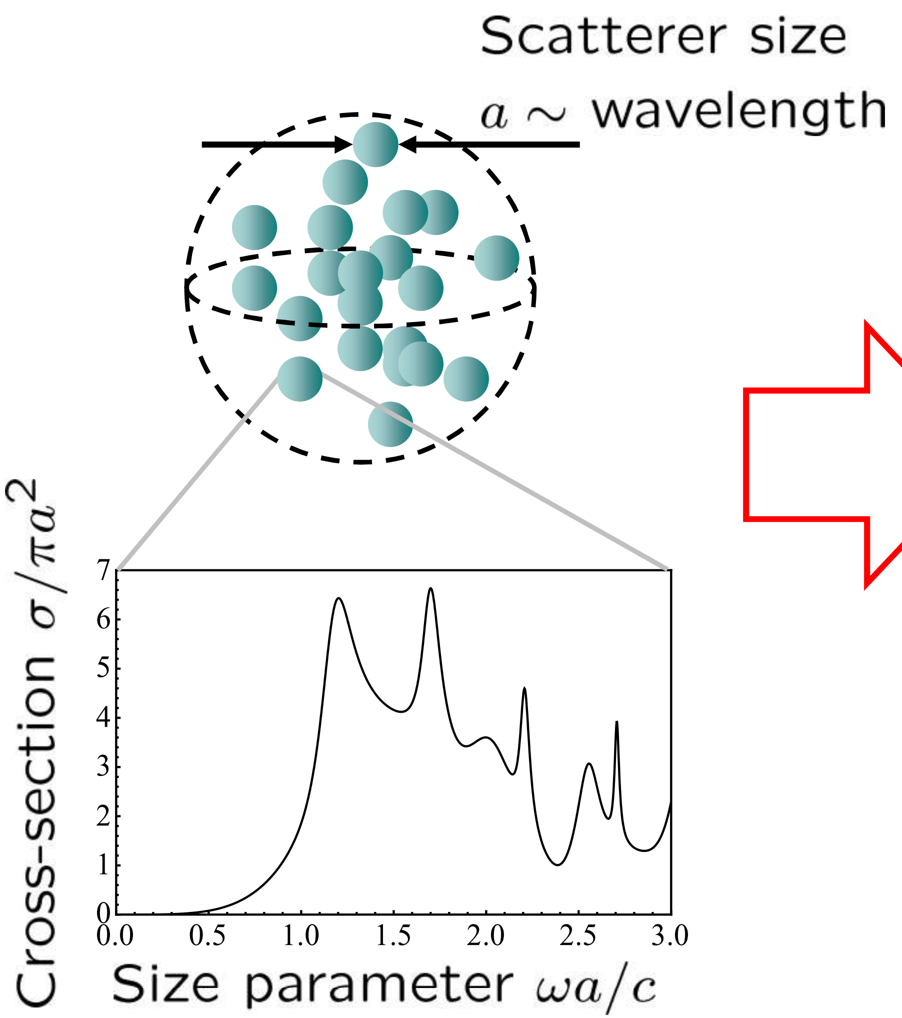
ξ — localization or correlation length

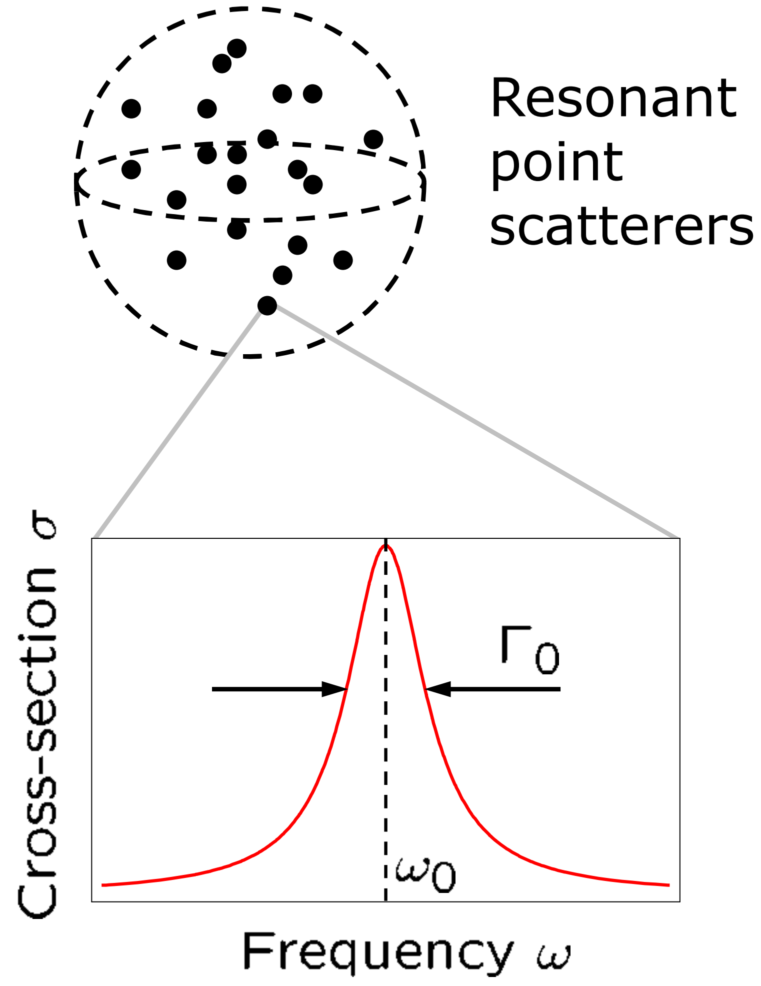
L — sample thickness

Cobus et al., PRL **116**, 193901 (2016)

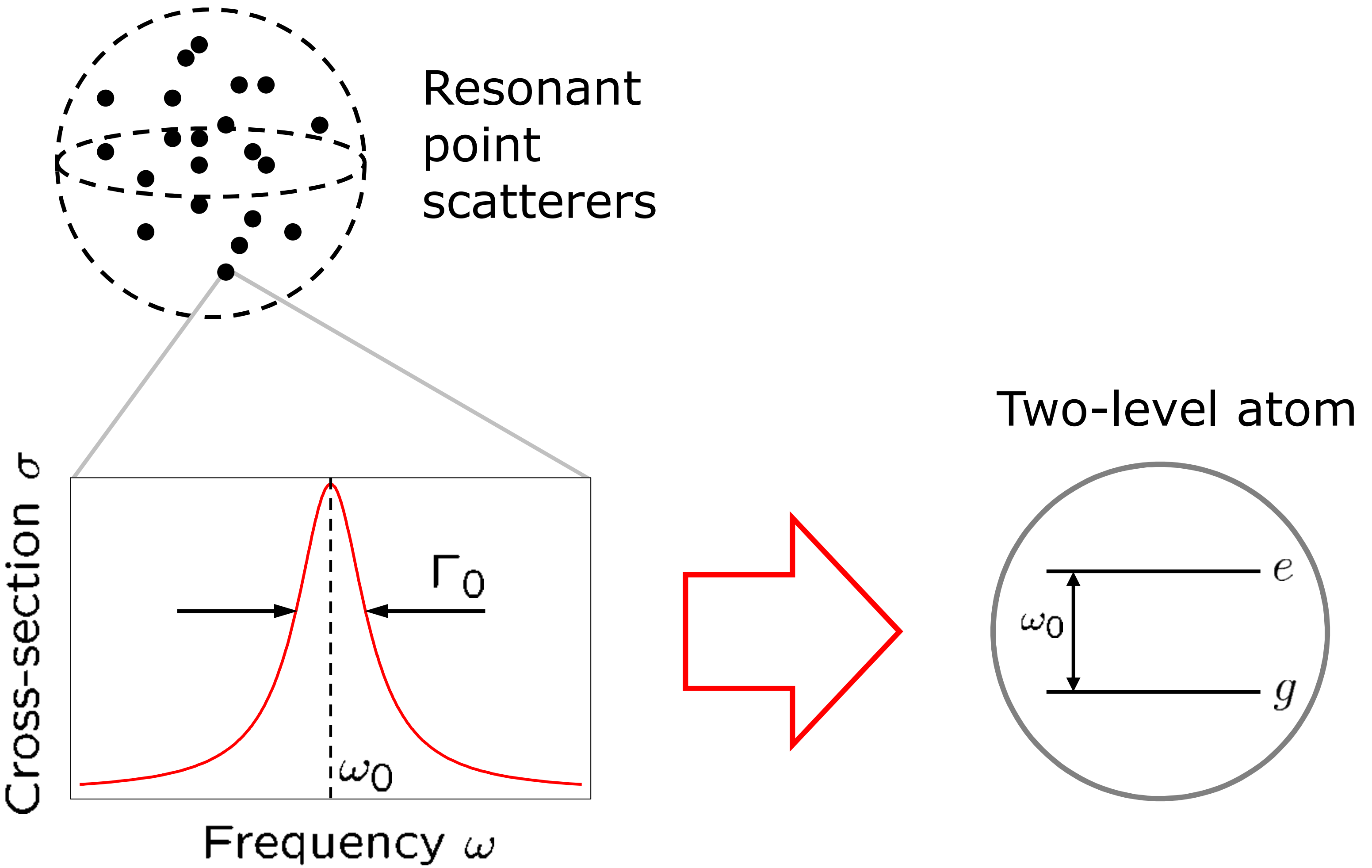
A minimal model of disordered media

Real samples complex | **Our model simpler**

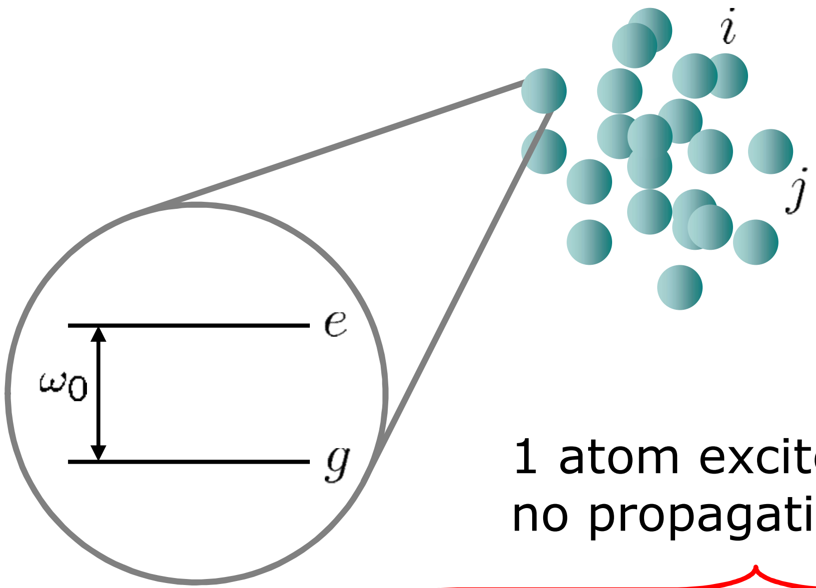




Atom is a realistic resonant point scatterer



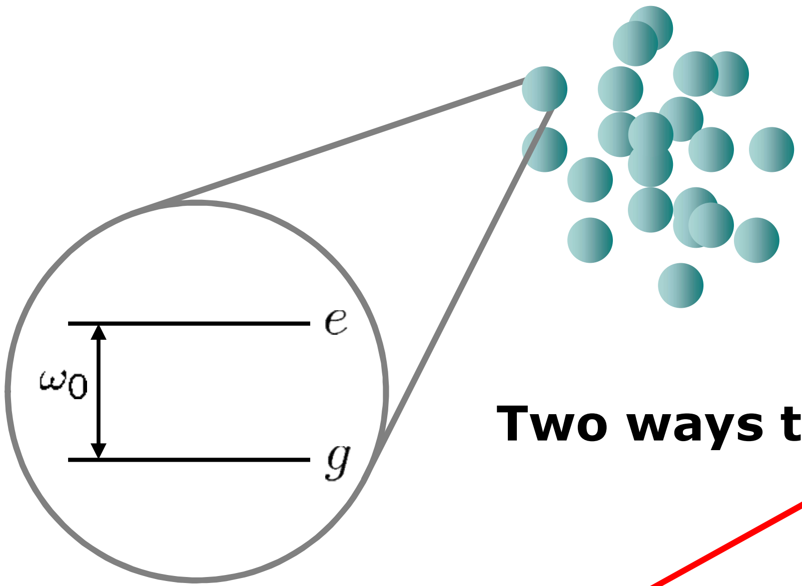
A scalar photon in a cloud of atoms



$$|\psi(t)\rangle = \overbrace{\sum_{j=1}^{N} \beta_j(t)|(N-1):g, j:e\rangle|0_R\rangle}^{\text{1 atom excited, no propagating photons}} + \overbrace{\sum_{\mathbf{k}} \gamma_{\mathbf{k}}(t)|N:g\rangle|\mathbf{k}\rangle}^{\text{all atoms in the ground state + a photon in the mode } \mathbf{k}}$$

$$+ \underbrace{\sum_{i<j}^{N} \sum_{\mathbf{k}} \alpha_{ij\mathbf{k}}(t)|(N-2):g, i:e, j:e\rangle|\mathbf{k}\rangle}_{\text{2 atoms excited + a (virtual) photon in the mode } \mathbf{k}}$$

A scalar photon in a cloud of atoms



Two ways to deal with the problem

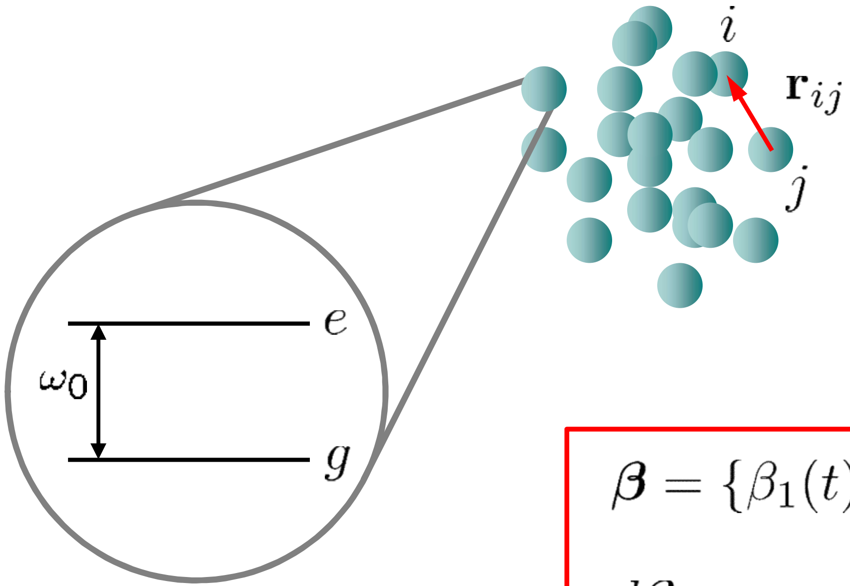
Waves (light) scattered
by the scatterers (atoms)

Equations to analyze:
wave equation with
random material properties
(dielectric constant)

Scatterers (atoms) coupled
by waves (light)

Equations to analyze:
dynamic equations for
identical nodes with
random couplings

Green's matrix



Green's matrix G describes propagation of light between pairs of atoms

$r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$

$$\boldsymbol{\beta} = \{\beta_1(t), \beta_2(t), \ldots, \beta_N(t)\}$$

$$\frac{d\boldsymbol{\beta}}{dt} = iG\boldsymbol{\beta}(t)$$

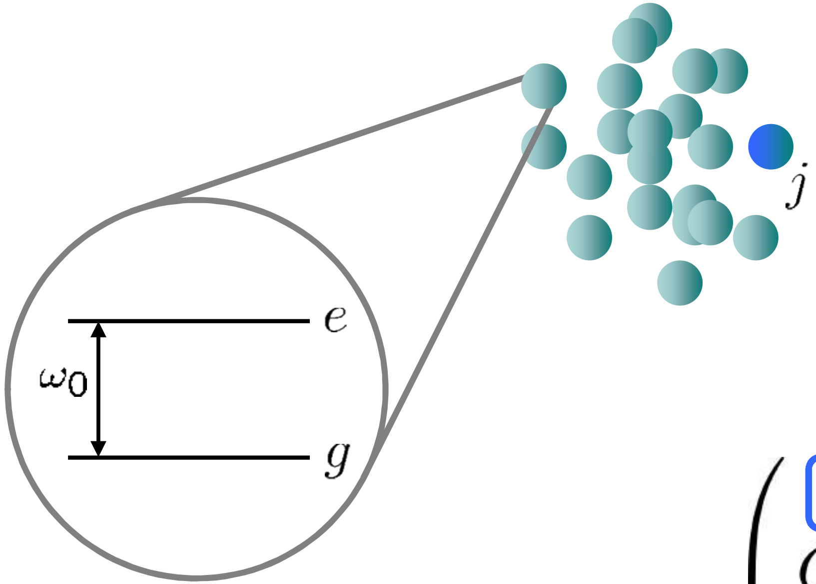
Euclidean random matrix

$$G_{ij} = i\delta_{ij} + (1 - \delta_{ij})\frac{e^{ik_0 r_{ij}}}{k_0 r_{ij}}$$

$$k_0 = \frac{\omega_0}{c}$$

Prasad & Glauber, PRA **31**, 1583 (1985)
Svidzinsky & Chang, PRA **77**, 043833 (2008)
Friedberg & Manassah, Phys. Lett. A **372**, 2514 (2008)

Structure of the Green's matrix

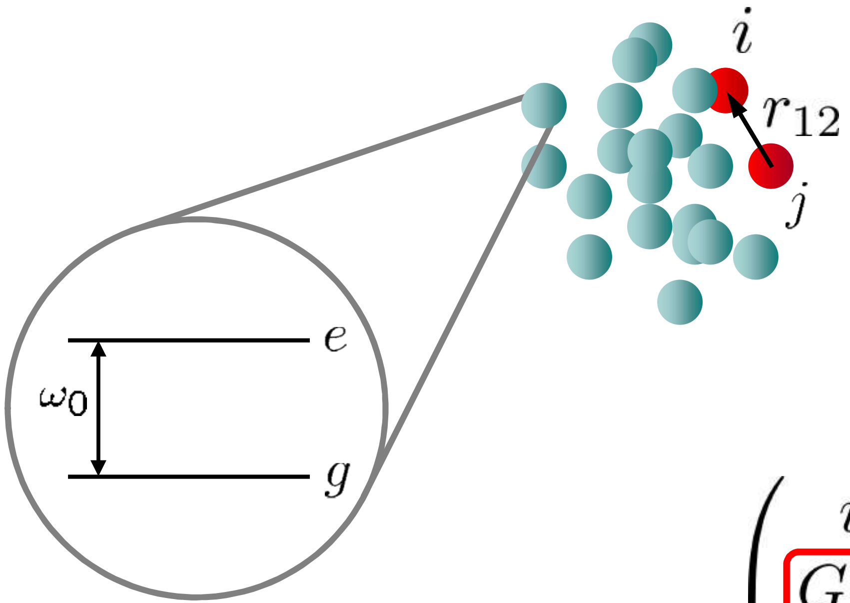


$$\begin{pmatrix} \boxed{i} & G_{12} & G_{13} & \dots & G_{1N} \\ G_{21} & \boxed{i} & G_{23} & \dots & G_{2N} \\ \dots & \dots & \dots & \dots & \dots \\ G_{N1} & G_{N2} & G_{N3} & \dots & \boxed{i} \end{pmatrix}$$

One-atom dynamics:

$G_{jj} = i$ describes the decay $e^{-\Gamma_0 t}$ of the excitation of an isolated excited atom

Structure of the Green's matrix

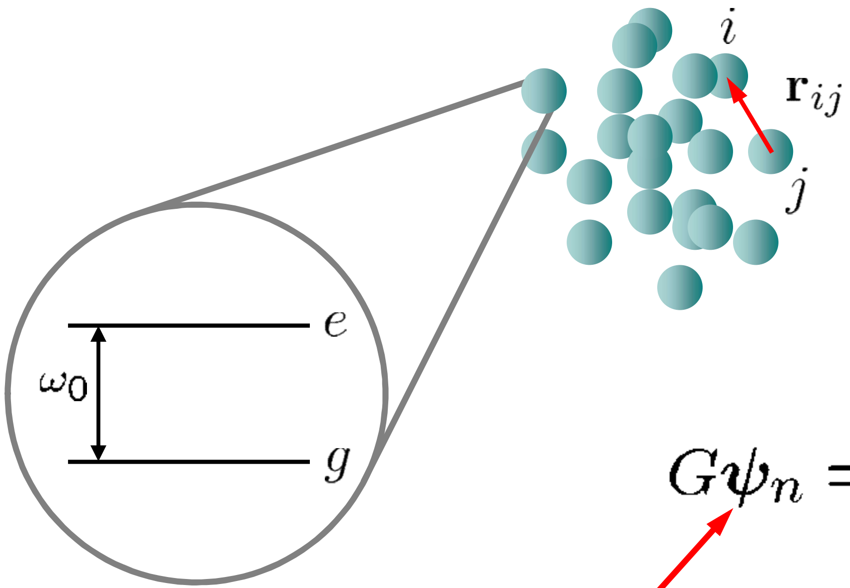


$$\begin{pmatrix} i & \boxed{G_{12}} & G_{13} & \dots & G_{1N} \\ \boxed{G_{21}} & i & G_{23} & \dots & G_{2N} \\ \dots & \dots & \dots & \dots & \dots \\ G_{N1} & G_{N2} & G_{N3} & \dots & i \end{pmatrix}$$

Pairwise coupling between atoms 1 & 2:

$G_{12} = e^{ik_0 r_{12}}/k_0 r_{12}$ is the field at position 2 due to a source at position 1

Quasi-modes of the system



Green's matrix G describes propagation of light between pairs of atoms

$r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$

$$G\boldsymbol{\psi}_n = \Lambda_n \boldsymbol{\psi}_n$$

Eigenvectors:

$$\boldsymbol{\psi}_n = \{\psi_n^1, \psi_n^2, \ldots, \psi_n^N\}$$

Eigenvalues:

$$\Lambda_n = \mathrm{Re}\Lambda_n + i\mathrm{Im}\Lambda_n$$

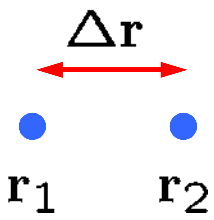
Frequency of the mode

Decay rate of the mode

Green's matrix for $N = 2$

$$G_{ij} = i\delta_{ij} + (1 - \delta_{ij})\frac{e^{ik_0|\mathbf{r}_i - \mathbf{r}_j|}}{k_0|\mathbf{r}_i - \mathbf{r}_j|}$$

$$\hat{G}\psi_n = \Lambda_n \psi_n$$



$$\hat{G} = \begin{pmatrix} i & \dfrac{e^{ik_0\Delta r}}{k_0\Delta r} \\ \dfrac{e^{ik_0\Delta r}}{k_0\Delta r} & i \end{pmatrix}$$

$$\boxed{\begin{aligned} \Lambda_{1,2} &= i \pm G_{12} \\ \psi_{1,2} &= (\pm 1, 1) \end{aligned}}$$

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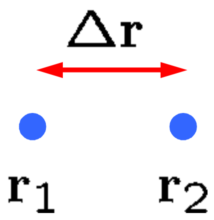

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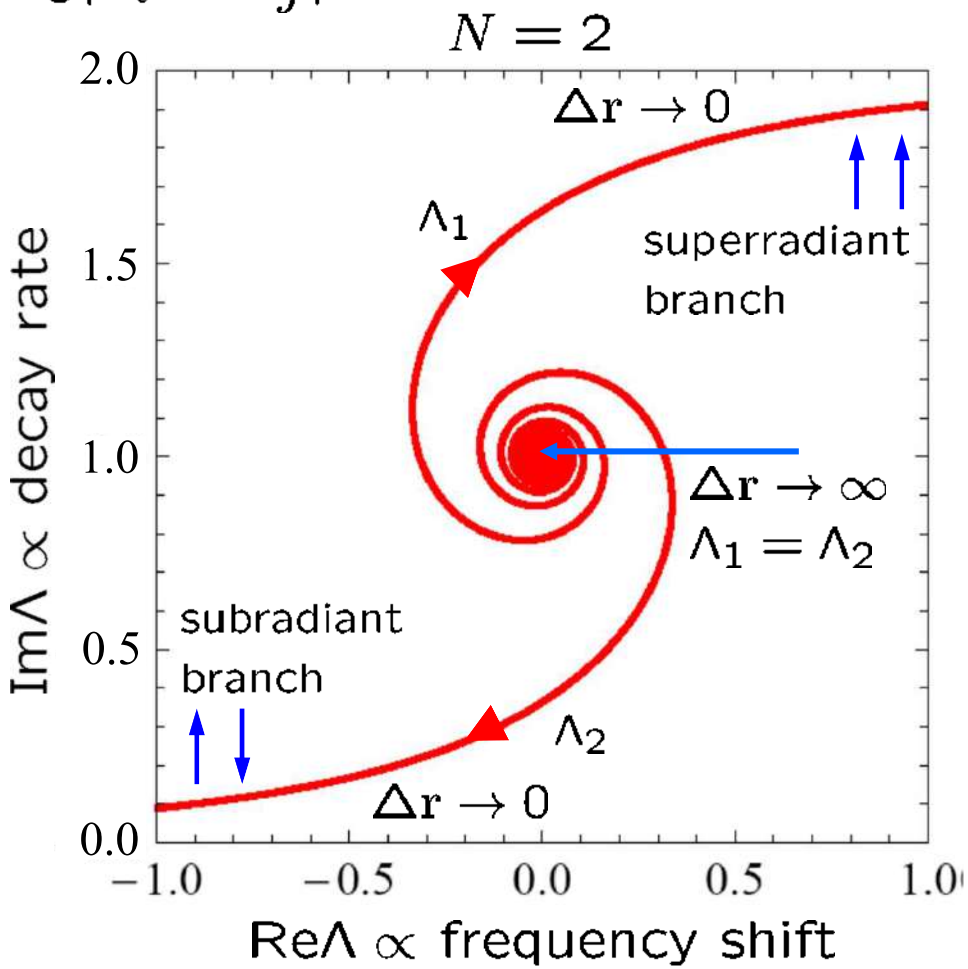
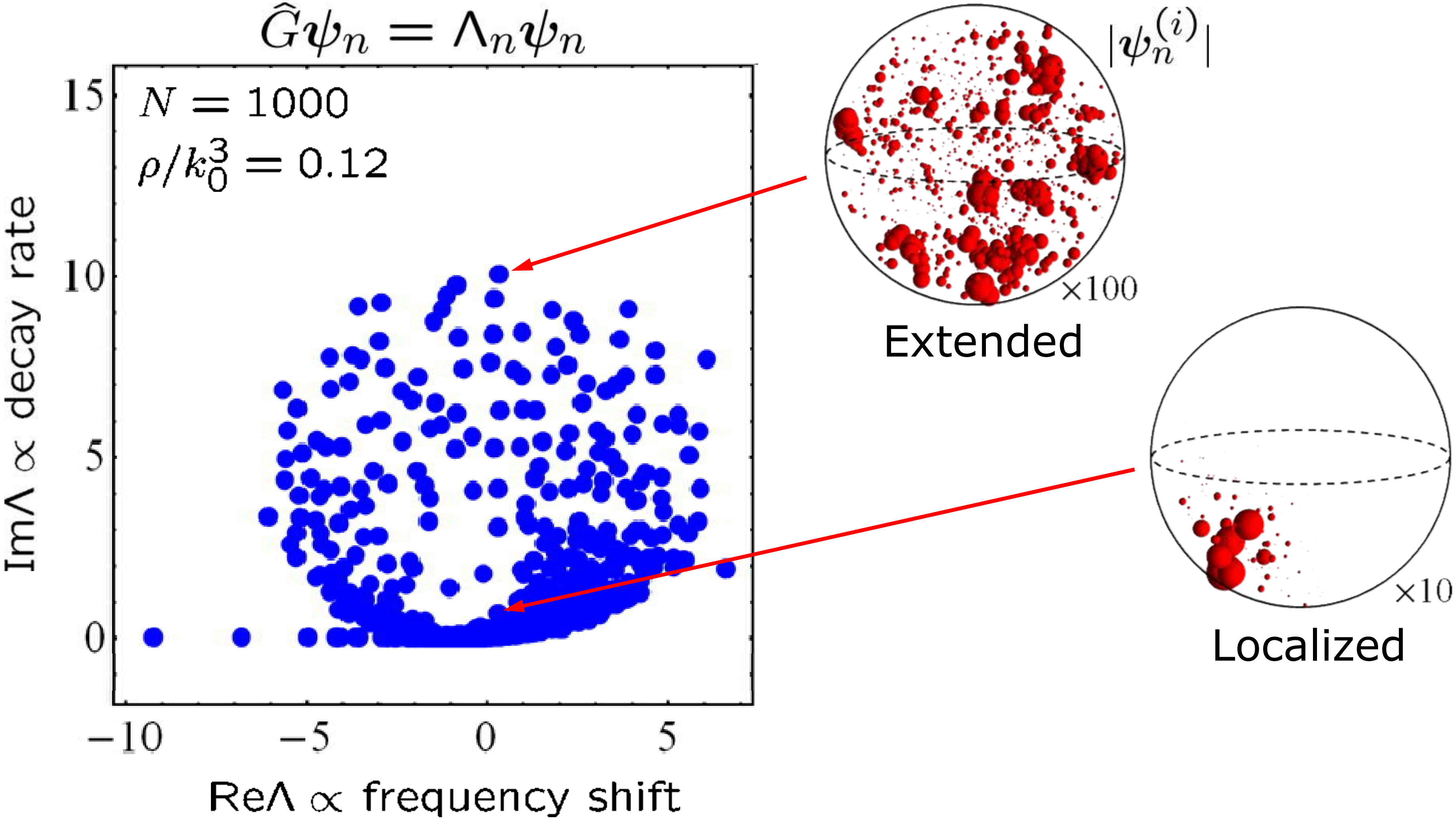
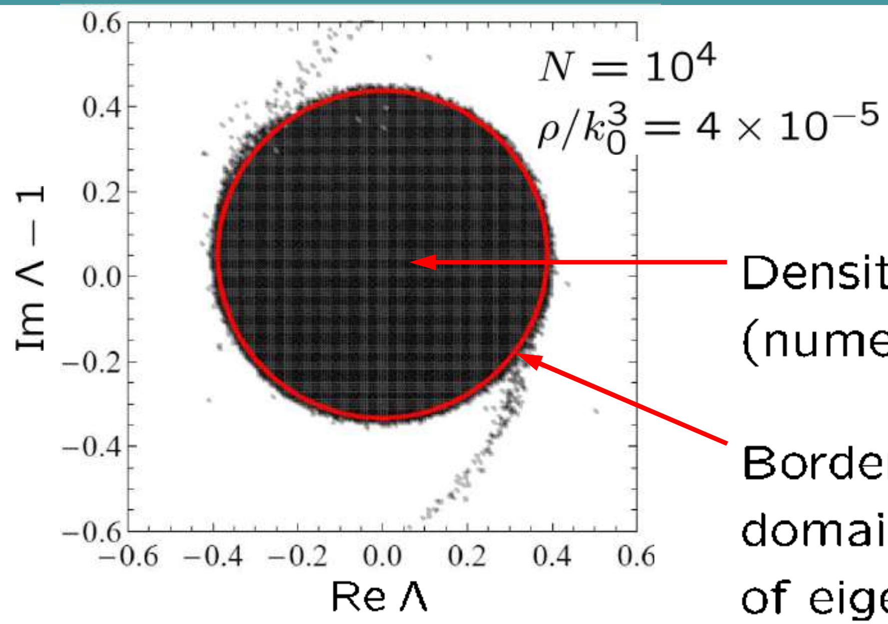


Green's matrix for $N \gg 1$



Goetschy & Skipetrov, EPL **96**, 34005 (2011)

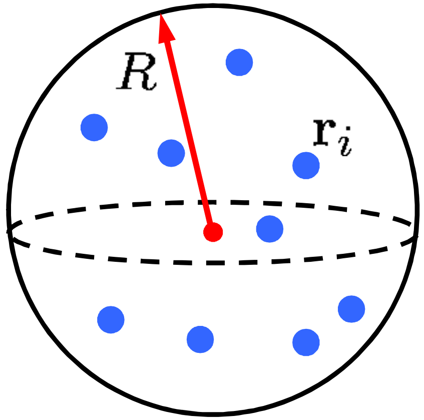
Eigenvalue density for $N \gg 1$



Density of eigenvalues of $\hat{G}$ (numerical)

Borderline of the domain of existence of eigenvalues of $\hat{G}$ (analytical)

$N \gg 1$ points in a sphere:

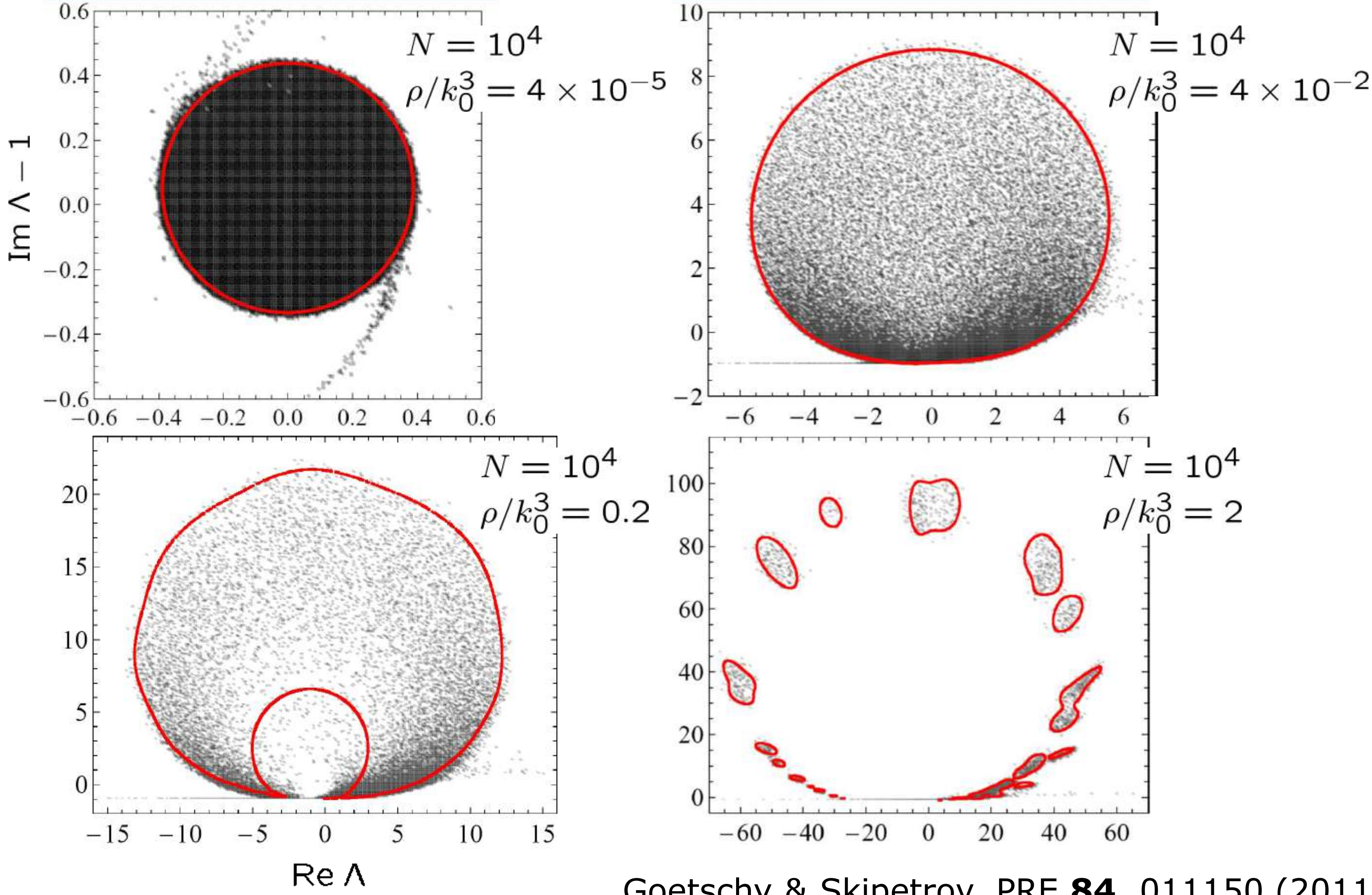


$$G_{ij} = i\delta_{ij} + (1 - \delta_{ij})\frac{e^{ik_0|\mathbf{r}_i - \mathbf{r}_j|}}{k_0|\mathbf{r}_i - \mathbf{r}_j|}$$

$$\hat{G}\psi_n = \Lambda_n \psi_n$$

Goetschy & Skipetrov, PRE **84**, 011150 (2011)

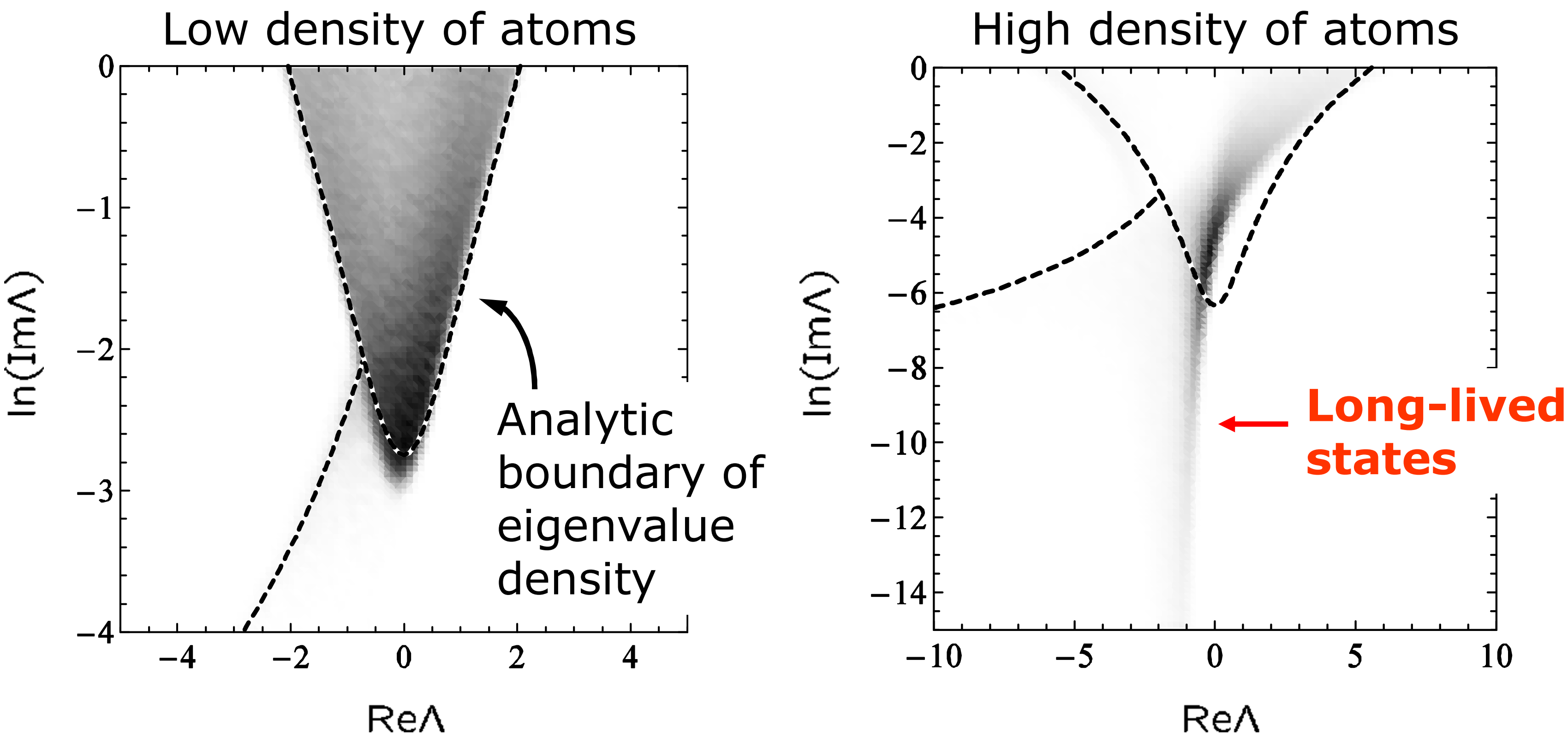
Eigenvalue density for $N \gg 1$



Goetschy & Skipetrov, PRE **84**, 011150 (2011)

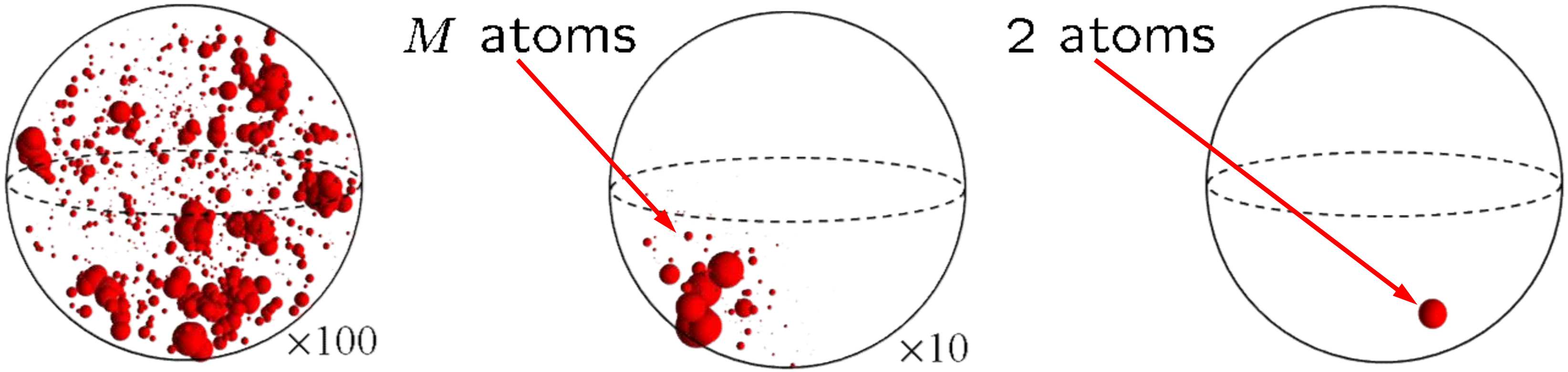
Breakdown of analytic theory for $\mathrm{Im}\,\Lambda \ll 1$

Density of eigenvalues



Skipetrov & Sokolov, PRL **112**, 023905 (2014)

Inverse participation ratio (IPR)

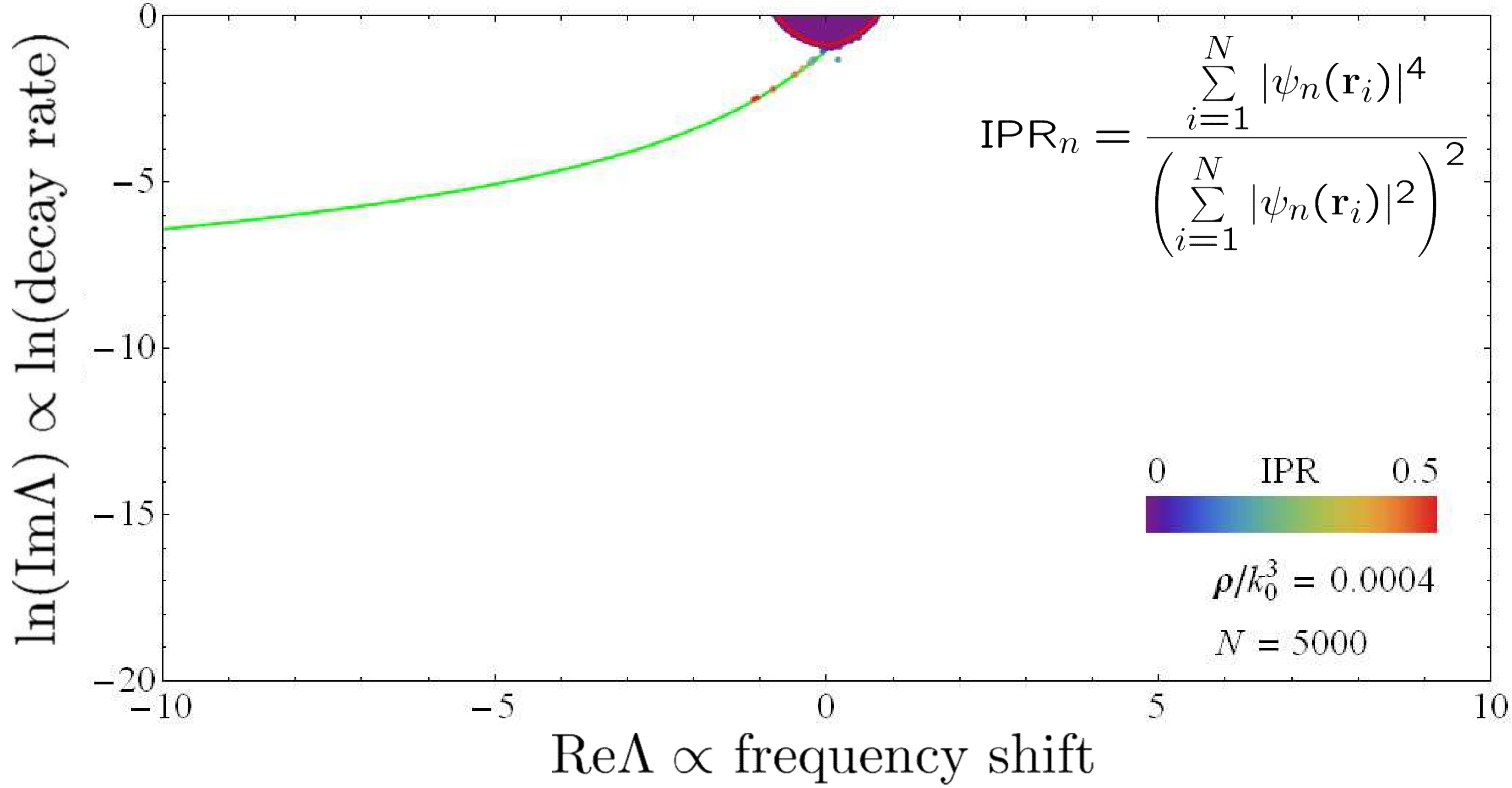


Small IPR	Appreciable IPR	Large IPR
$\sim 1/N \ll 1$	$\sim 1/M \gg 1/N$	$\sim 1/2$

$$\mathrm{IPR}_n = \frac{\sum_{i=1}^{N} |\psi_n(\mathbf{r}_i)|^4}{\left(\sum_{i=1}^{N} |\psi_n(\mathbf{r}_i)|^2\right)^2} \sim \frac{1}{M}$$

for a state (mode) localized on M atoms

Evolution of IPR with increasing density

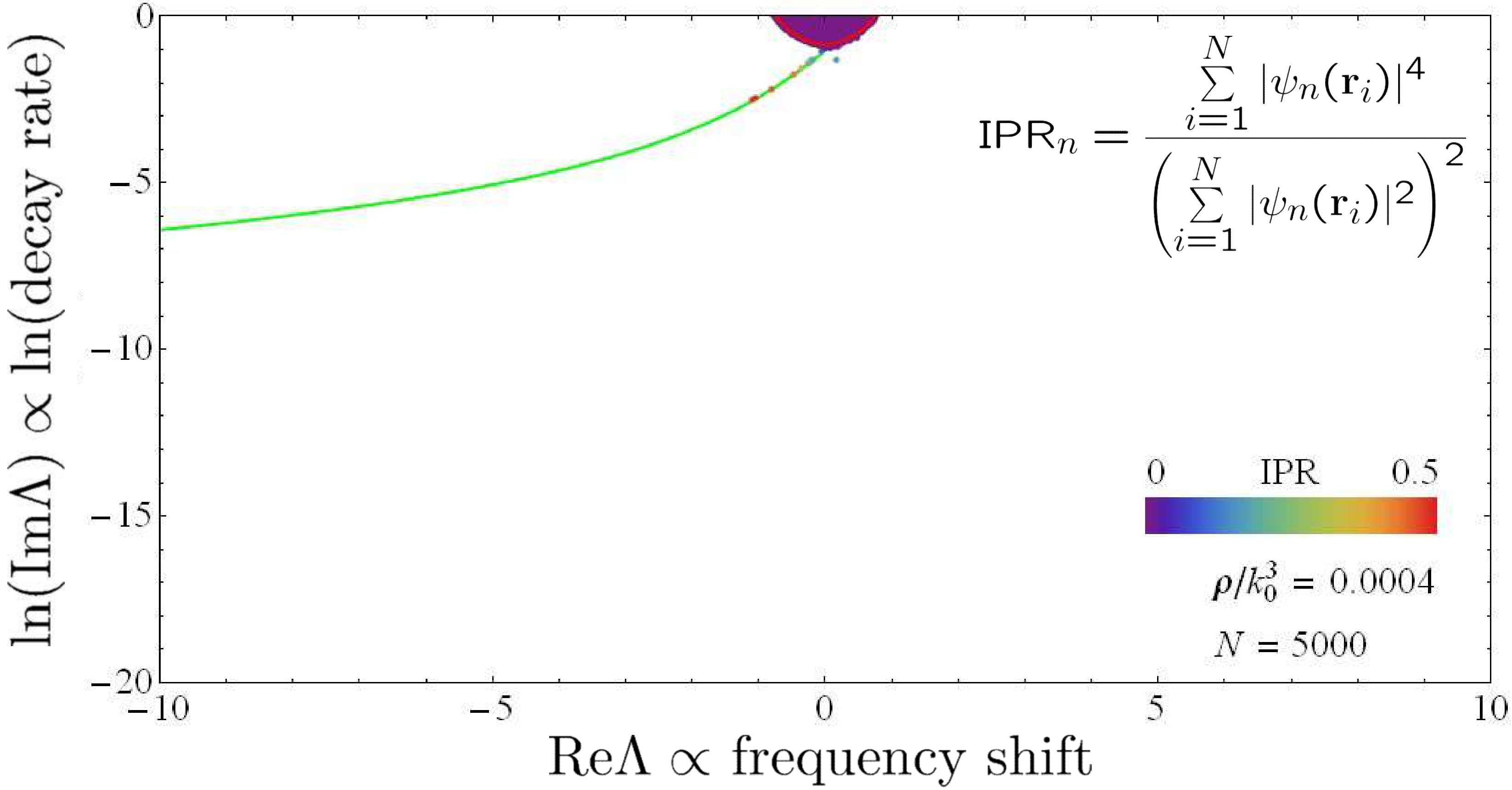


Eigenvalue domain boundary from the diffusion theory

Subradiant states localized on 2 closely located atoms

Skipetrov & Sokolov, PRL **112**, 023905 (2014)

Evolution of IPR with increasing density



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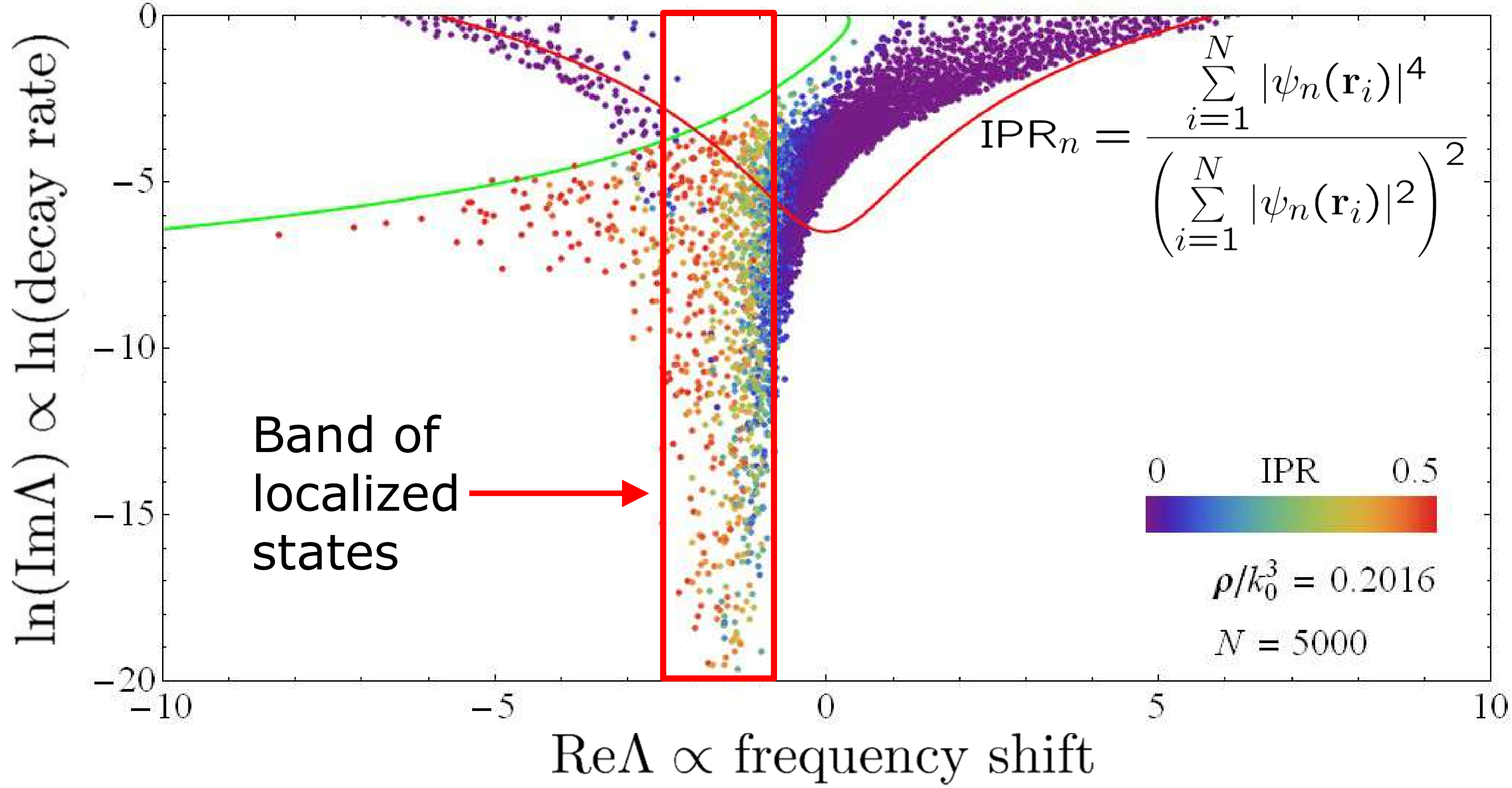


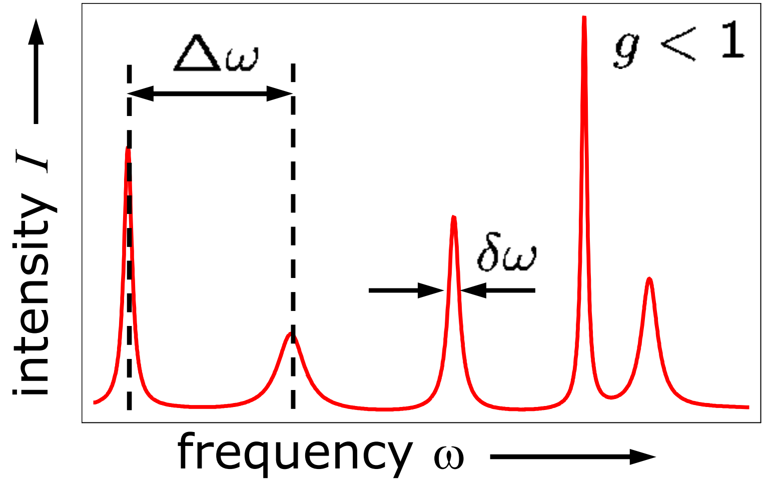
Eigenvalue domain boundary from the diffusion theory

Subradiant states localized on 2 closely located atoms

Skipetrov & Sokolov, PRL **112**, 023905 (2014)

Dimensionless conductance

Spectrum of a disordered system



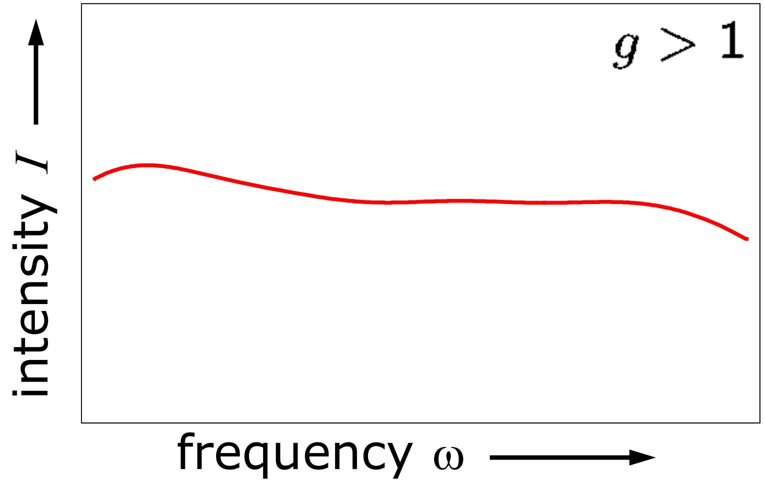
$\delta\omega \sim \mathrm{Im}\Lambda$ — "mode width"

$\Delta\omega \sim \mathrm{Re}\Lambda_{n+1} - \mathrm{Re}\Lambda_n$ — "mode spacing"

$g = \frac{\delta\omega}{\Delta\omega}$ - Thouless parameter

(dimensionless conductance)

The same with mode widths x 10



Thouless criterion

of Anderson localization:

$$g < 1$$

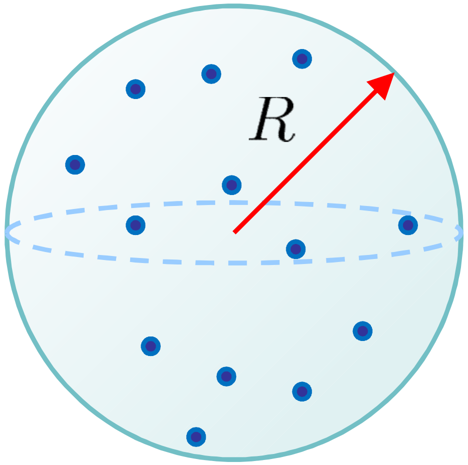
Scaling theory of Anderson localization

Main idea: Study how g evolves with sample size R

If the modes are extended, g grows with R

If the modes are localized, g decreases with R

At the critical point $g = g_c$ is independent of R



Abrahams et al., PRL **42**, 673 (1979)

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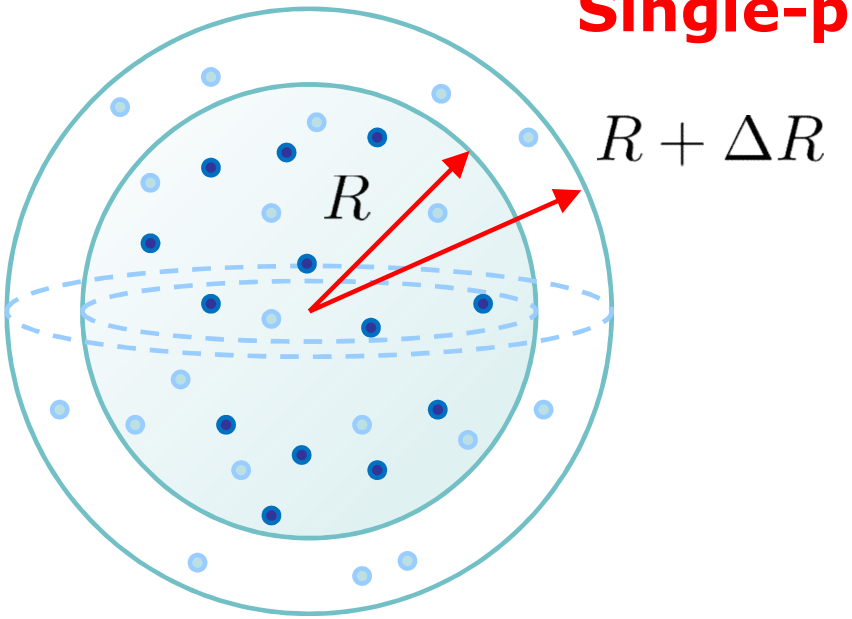


Single-parameter scaling hypothesis

$$\boxed{\beta(g) = \frac{\partial \ln g}{\partial \ln k_0 R}}$$

> 0 if extended eigenstates

< 0 if localized eigenstates

Abrahams et al., PRL **42**, 673 (1979)

Scaling theory of Anderson localization

Abrahams et al., PRL **42**, 673 (1979):

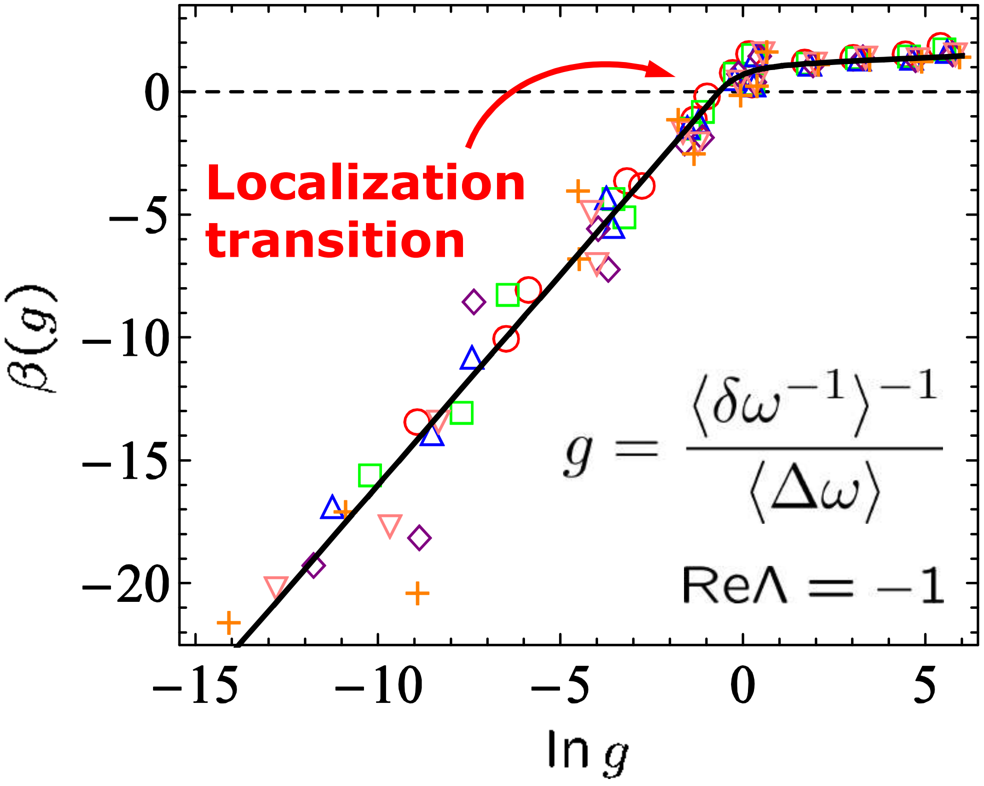
$$\boxed{\beta(g) = \frac{\partial \ln g}{\partial \ln k_0 R}}$$

> 0 if extended eigenstates

< 0 if localized eigenstates

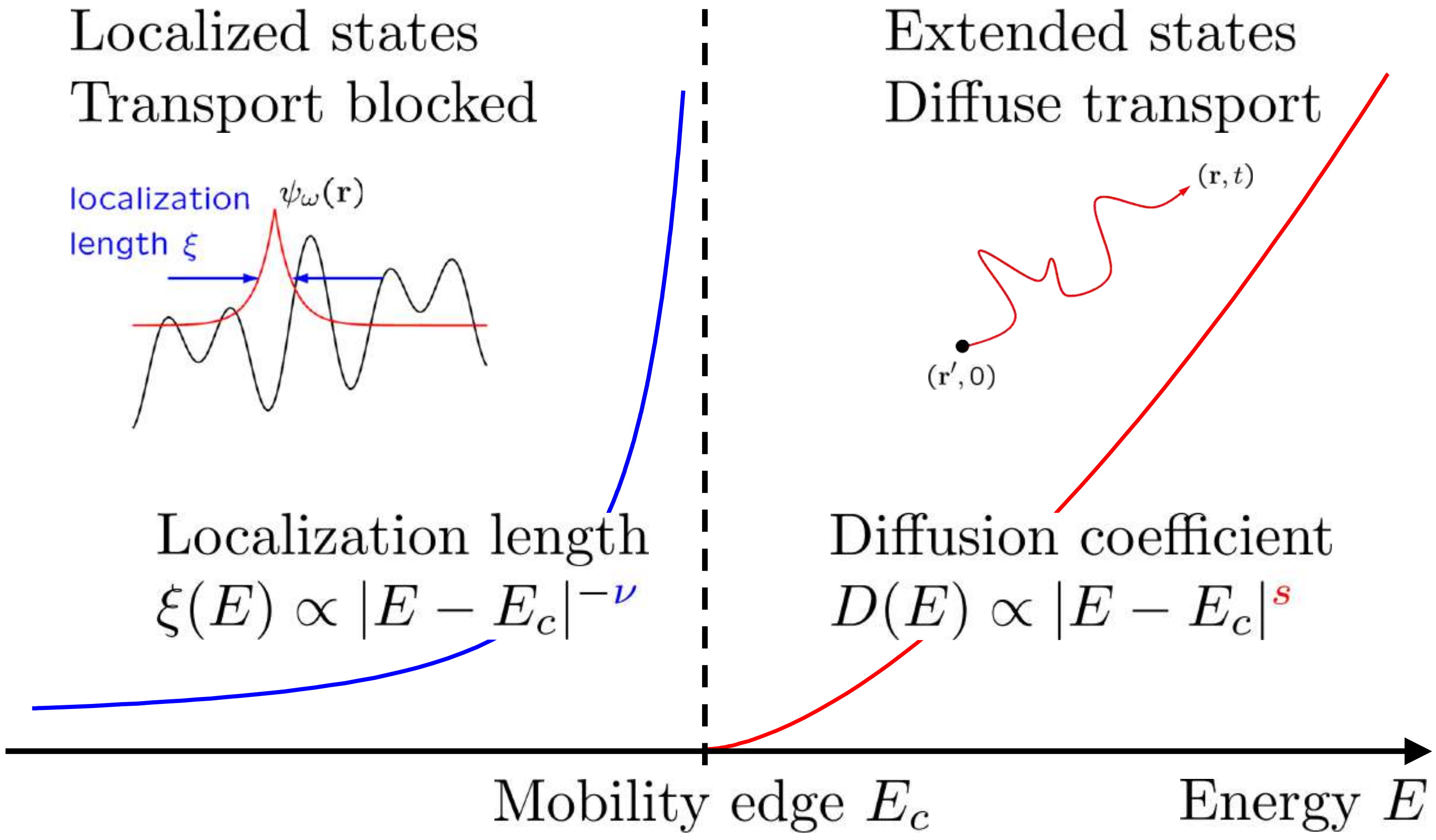
= 0 at the critical point

Random Green's matrix model (*d*=3):



Skipetrov & Sokolov, PRL **112**, 023905 (2014)

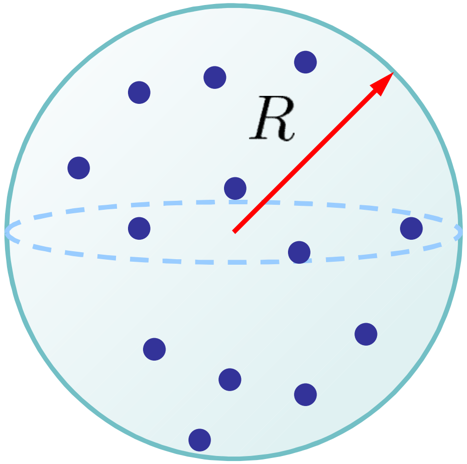
Critical behavior around the mobility edge



Critical exponents $s = \nu$ (in 3D)

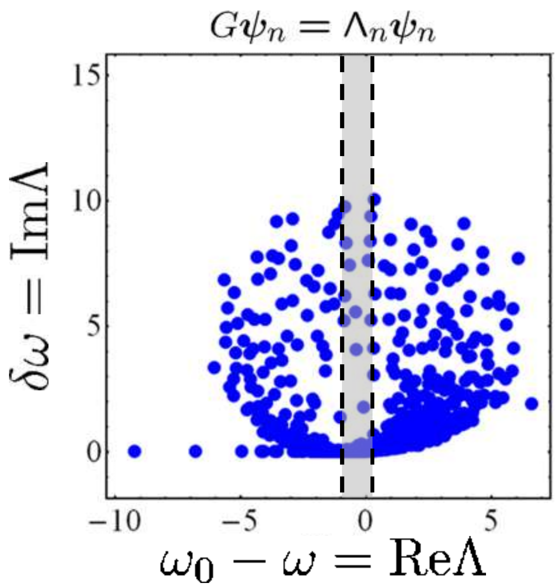
Thouless conductance & scaling

N scatterers at density ρ



$$G_{ij} = i\delta_{ij} + (1-\delta_{ij})\frac{e^{ik_0 r_{ij}}}{k_0 r_{ij}}$$

Eigenvalues of the Green's matrix



Thouless conductance

$$\delta\omega = \mathrm{Im}\Lambda$$

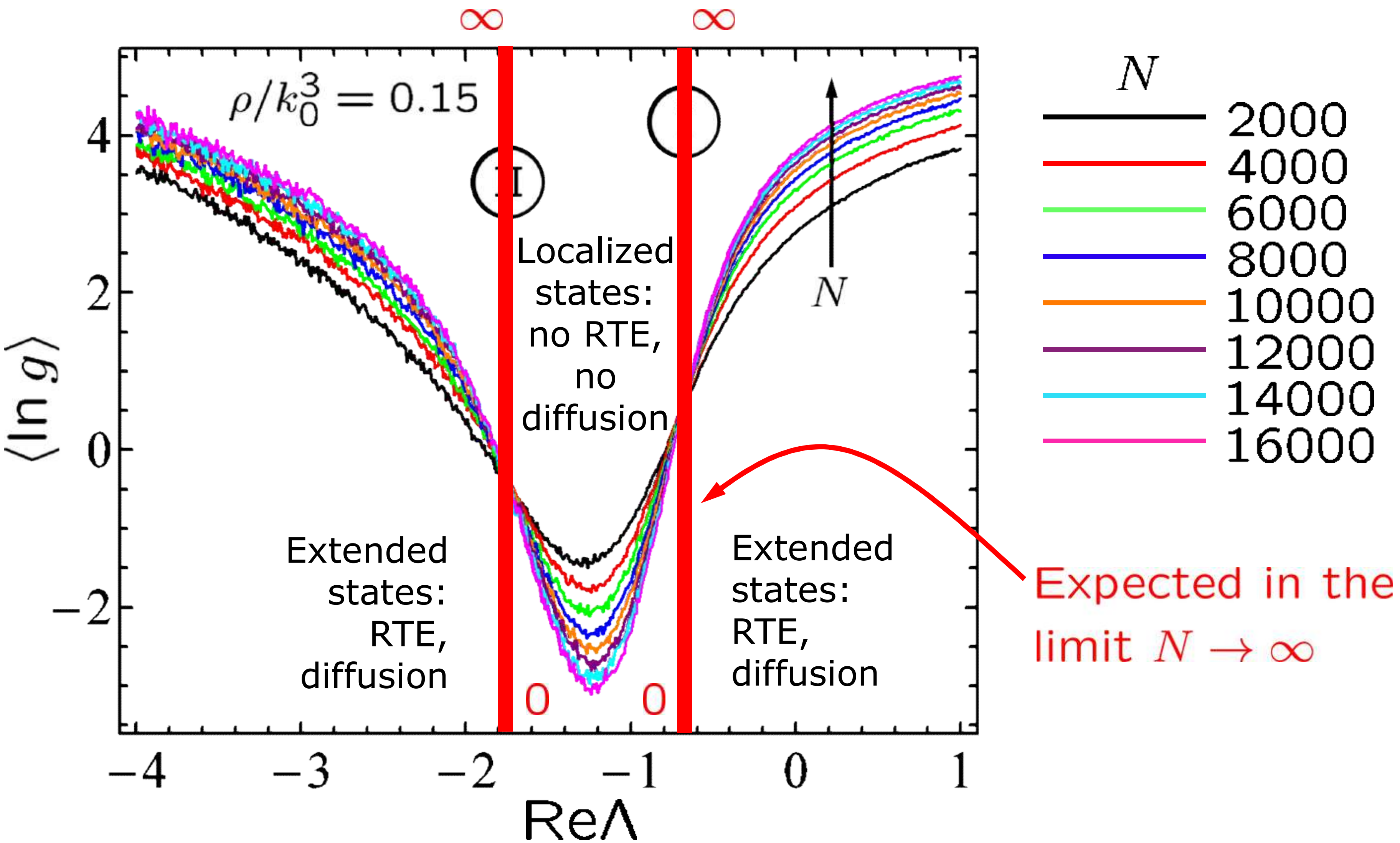
$$\boxed{g(\omega) = \frac{\delta\omega}{\langle \Delta\omega \rangle}}$$

$$\langle \Delta\omega \rangle = \langle \omega_{n+1} - \omega_n \rangle$$

Now we are going to study statistical properties of $g(\omega)$ at high $\rho = 0.15 k_0^3$ at which localized states are expected

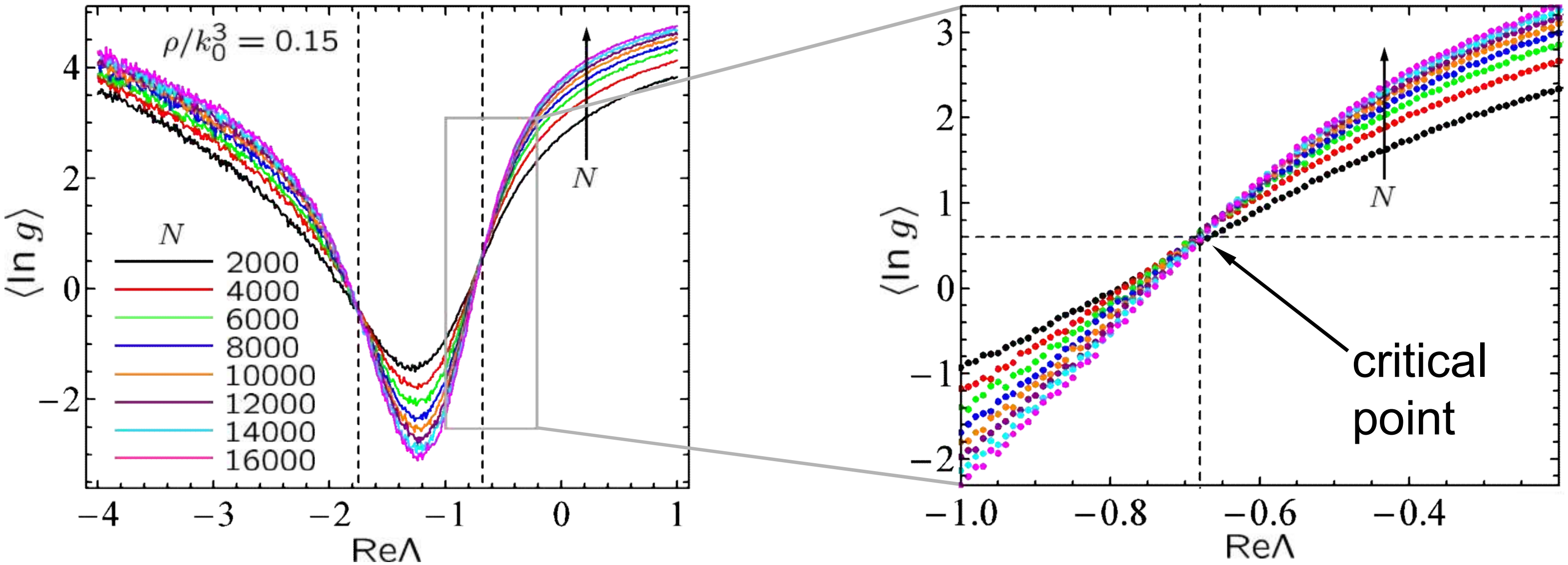
Skipetrov, PRB **94**, 064202 (2016)

Scaling of moments of ln g

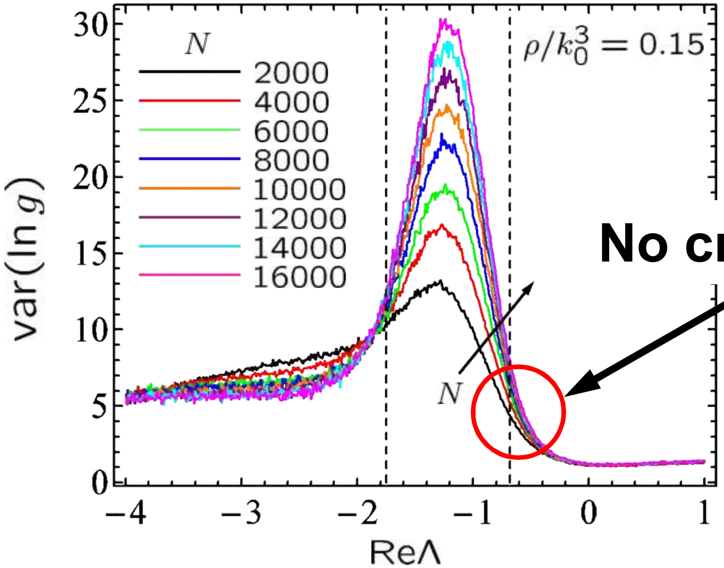


Skipetrov, PRB **94**, 064202 (2016)

Scaling of moments of ln *g*



But...

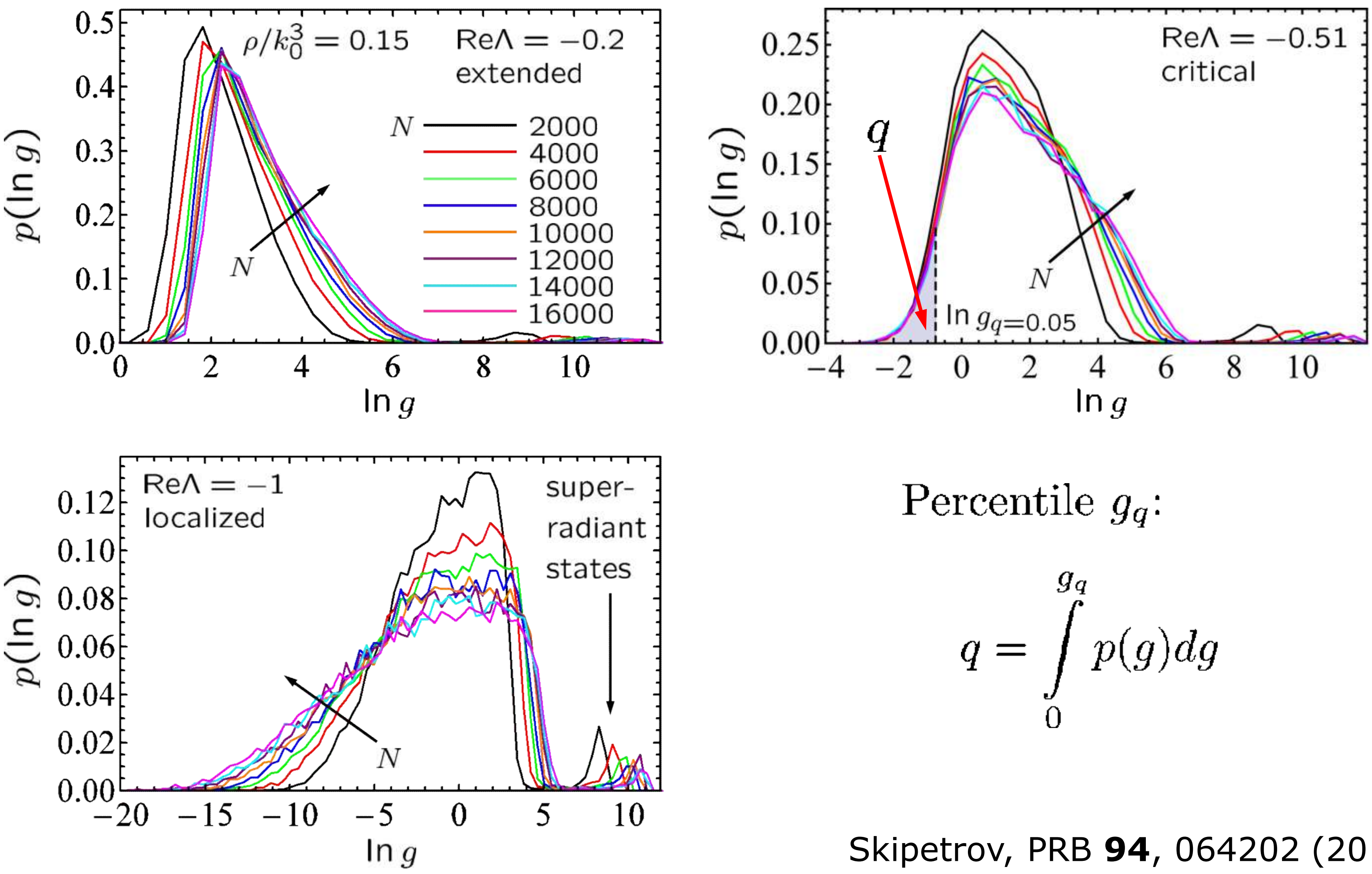


No crossing

No single-parameter scaling

Skipetrov, PRB **94**, 064202 (2016)

Distribution of conductance



Percentile g_q:

$$q = \int_0^{g_q} p(g)dg$$

Skipetrov, PRB **94**, 064202 (2016)

Single-parameter scaling

R — system size

$$\boxed{\ln g_q = F_q(R/\xi)}$$

$$\xi \propto \frac{1}{(\mathrm{Re}\Lambda - \mathrm{Re}\Lambda_c)^{\nu}}$$

— localization length

$$\begin{aligned}\ln g_q &= F_q(R/\xi) = F_q[R(\mathrm{Re}\Lambda - \mathrm{Re}\Lambda_c)^{\nu}] \\ &= F_q[R^{1/\nu}(\mathrm{Re}\Lambda - \mathrm{Re}\Lambda_c)] \longrightarrow F_q(\psi, \phi)\end{aligned}$$

Relevant scaling variable:

$$\psi = R^{1/\nu} u(\mathrm{Re}\Lambda - \mathrm{Re}\Lambda_c), \quad u(x) = u_1 x + u_2 x^2 + \ldots$$

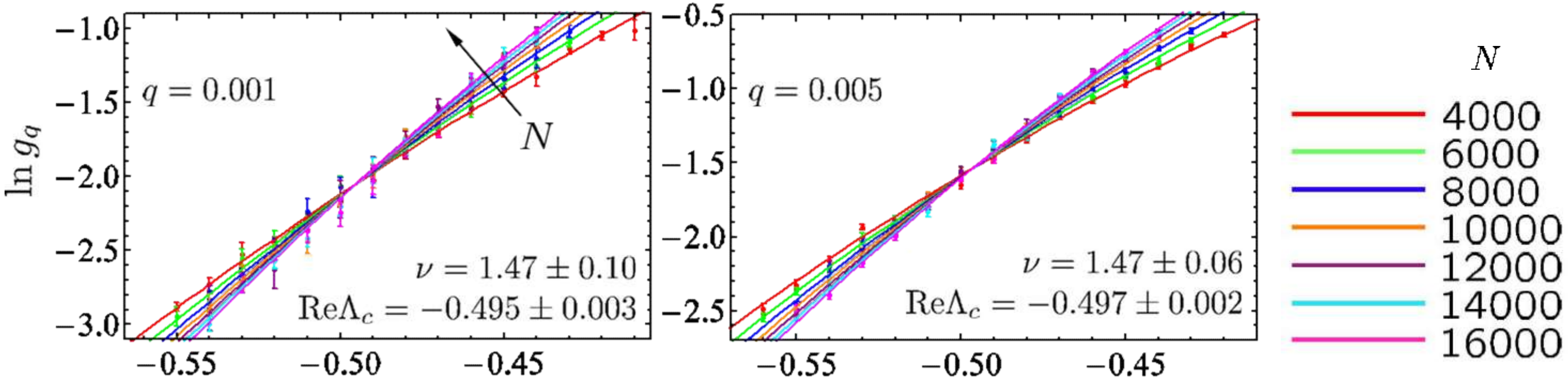
Irrelevant scaling variable:

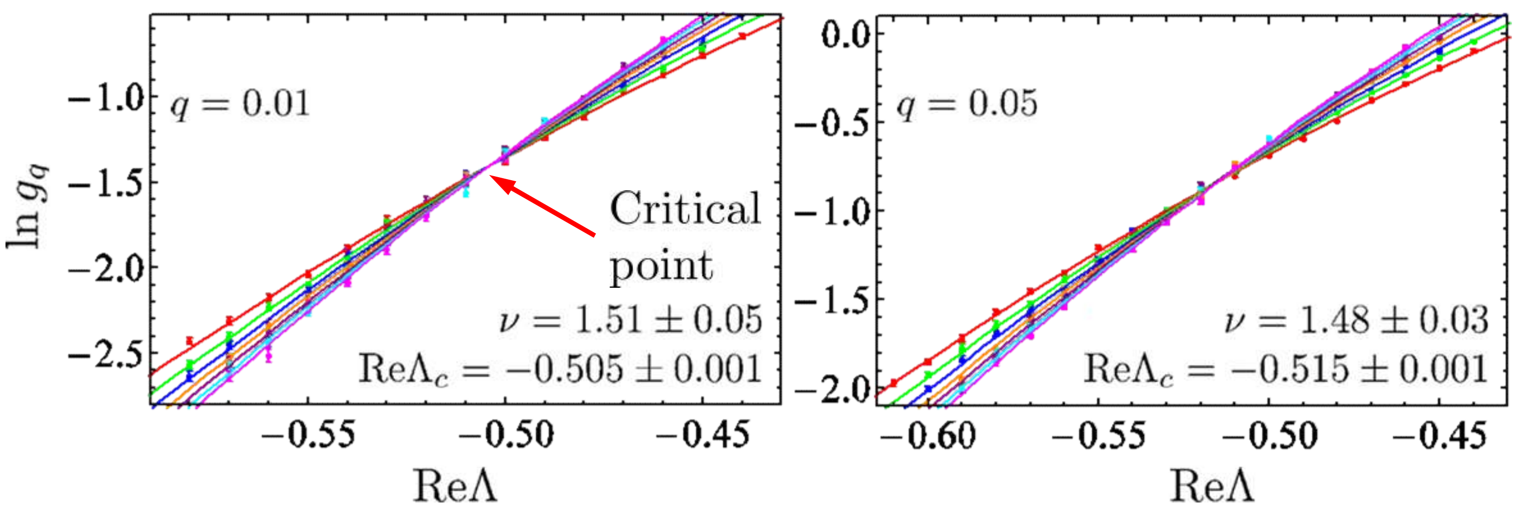
$$\phi = R^{-y} v(\mathrm{Re}\Lambda - \mathrm{Re}\Lambda_c), \quad v(x) = v_0 + v_1 x + v_2 x^2 + \ldots$$

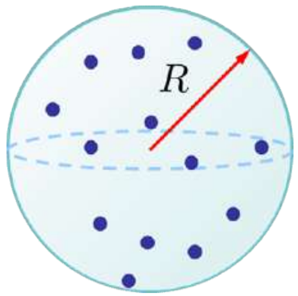
Slevin, Markos, Ohtsuki, PRB **67**, 155106 (2003)

Finite-size scaling of percentiles

$$\ln g_q = F_q(R/\xi), \quad \xi \propto (\mathrm{Re}\Lambda - \mathrm{Re}\Lambda_c)^{-\nu}$$

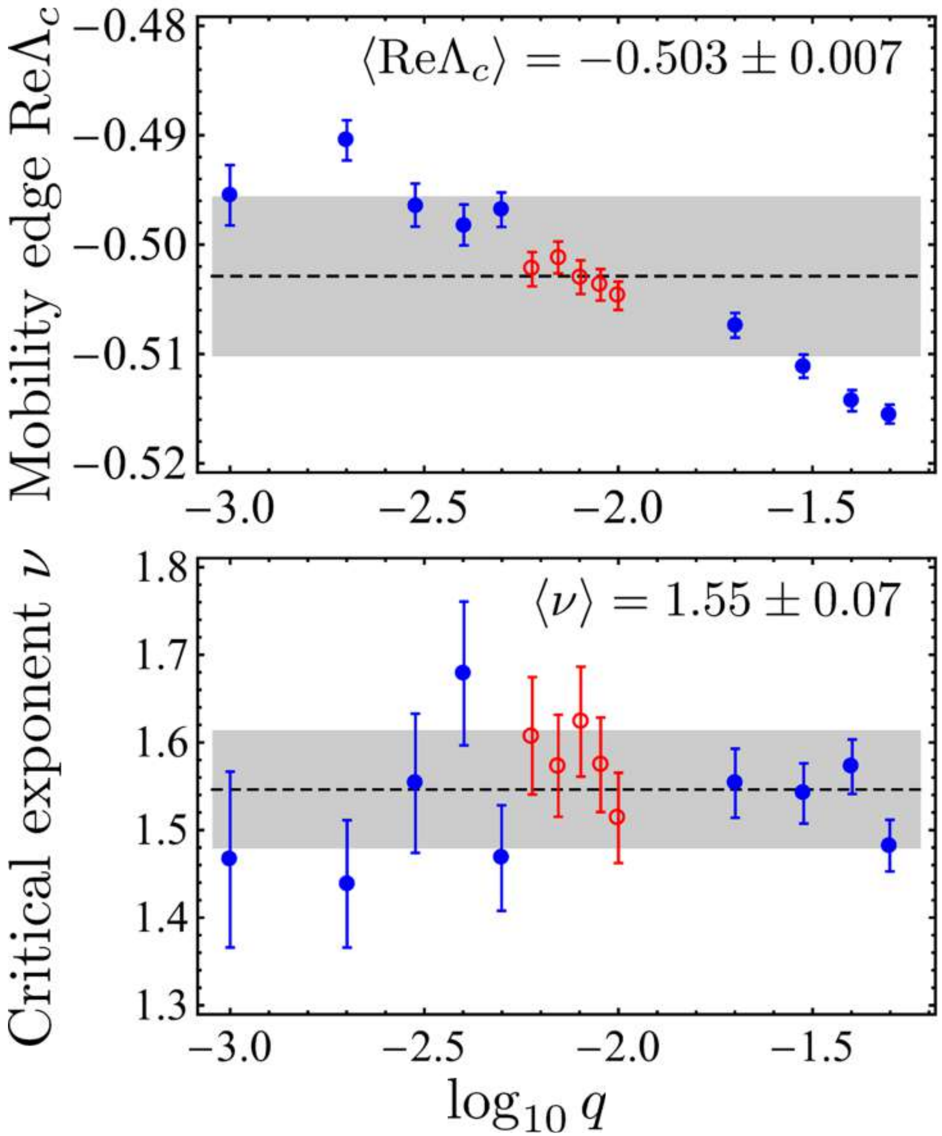






Skipetrov, PRB **94**, 064202 (2016)

Best-fit parameters

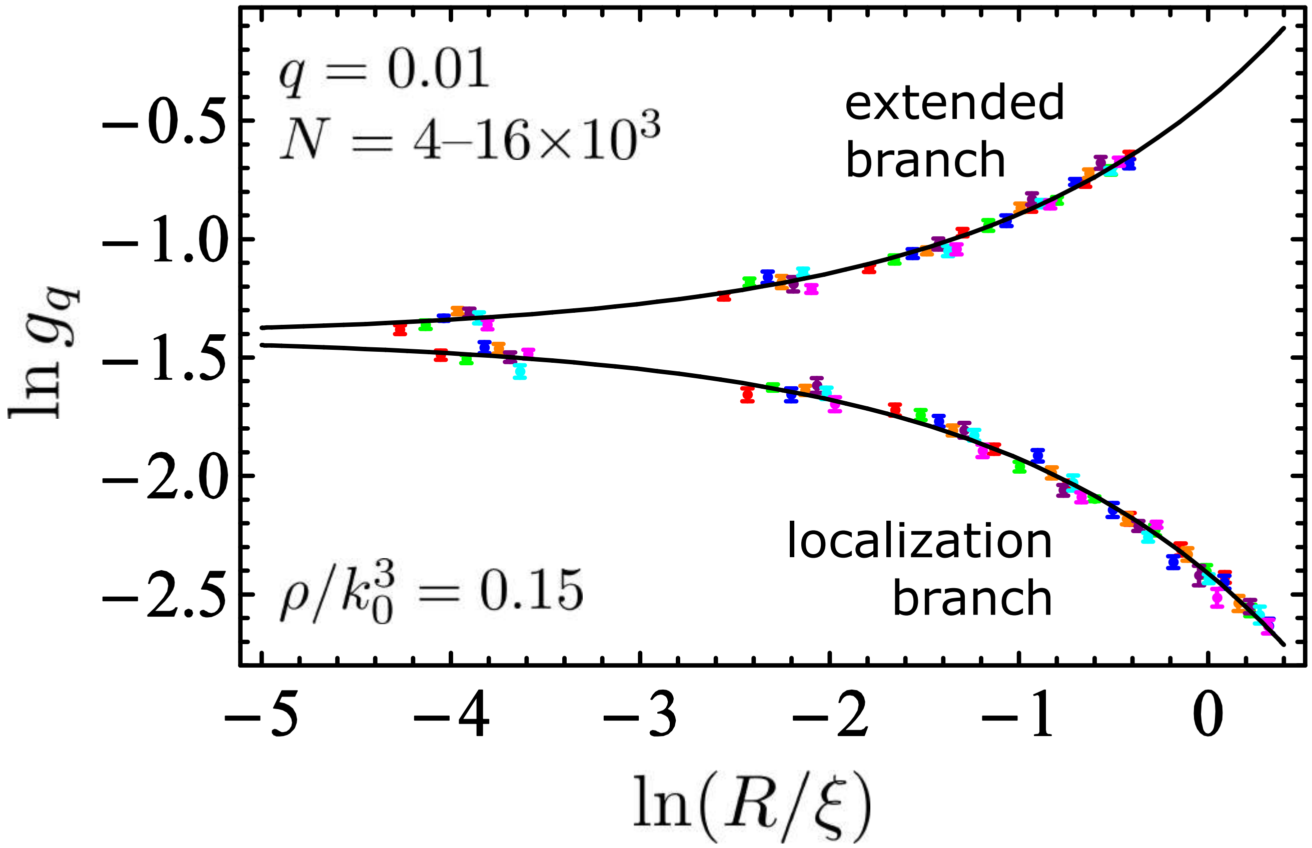


The value of critical exponent following from the fits is close to $\nu \simeq 1.57$ expected for the 3D orthogonal symmetry class.

We conclude that the observed transition is likely to belong to the same symmetry class as the **Anderson transition in a system of spinless electrons.**

Skipetrov, PRB **94**, 064202 (2016)

Single-parameter scaling: a posteriori check



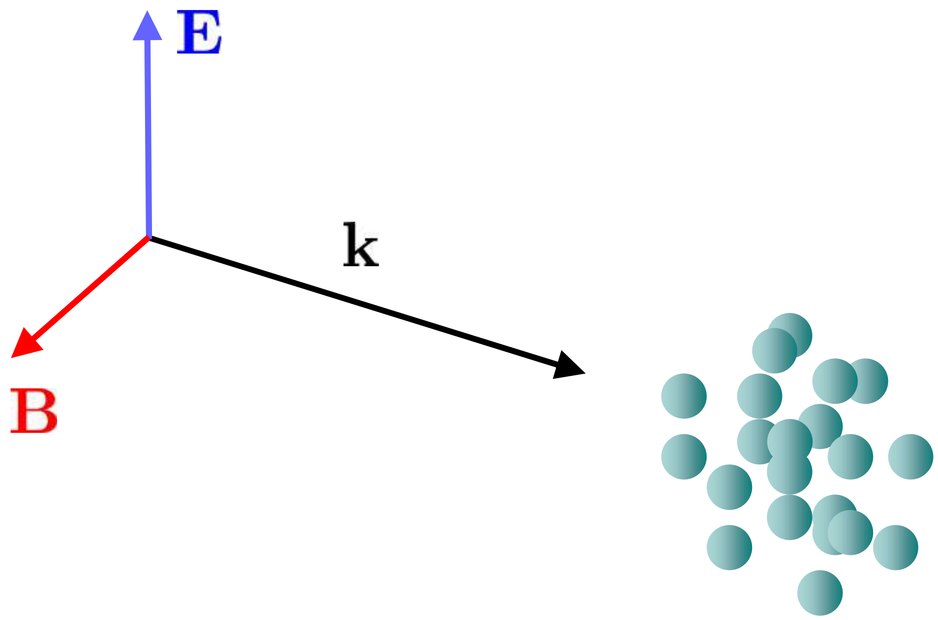
Data for different N all fall on a single line $\ln g_q = F_q(R/\xi)$

R — sample size

ξ — localization (correlation) length

Skipetrov, PRB **94**, 064202 (2016)

Anderson localization of light

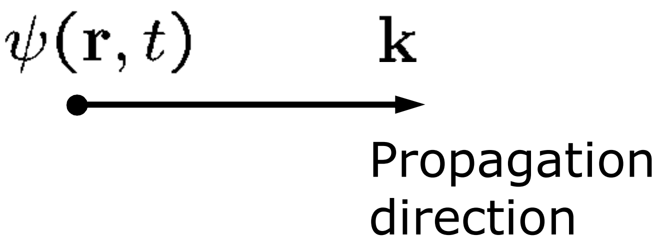


Pioneering theoretical works: John, PRL **53**, 2169 (1984)
Anderson, Philos. Mag. B **52**, 505 (1985)

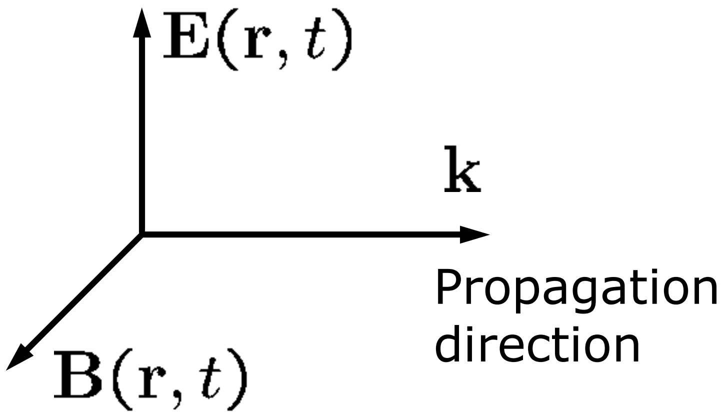
Experiments inconclusive: Wiersma et al., Nature **390**, 671 (1997)
Sperling et al., Nat. Photonics **7**, 48 (2013)

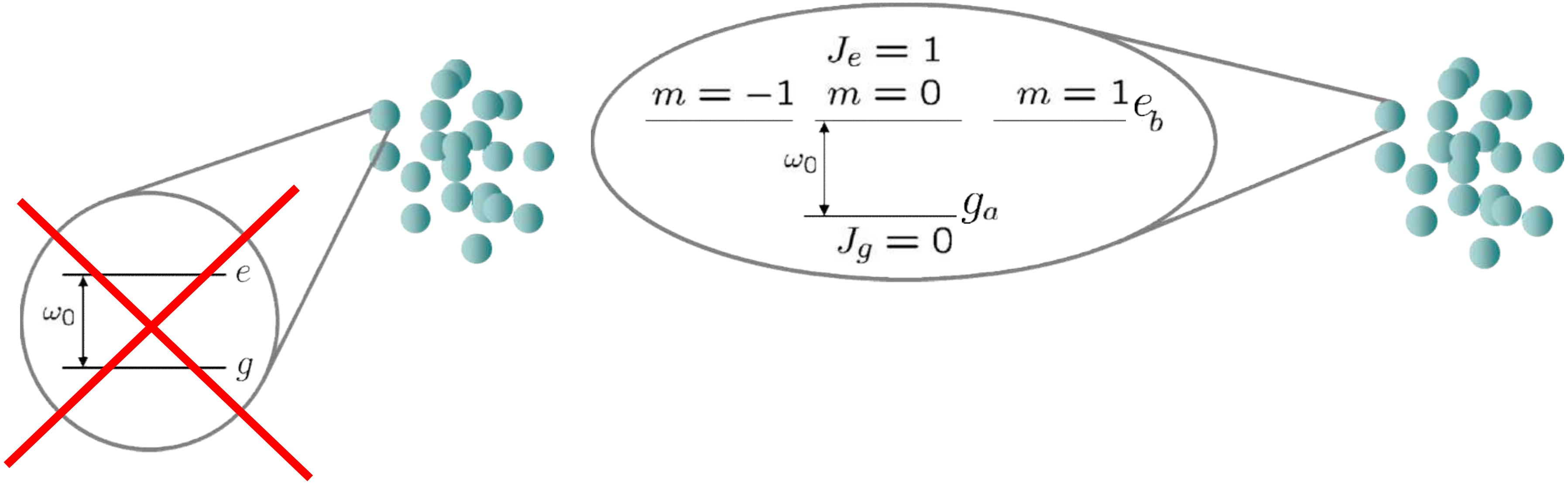
Light is a vector wave

"Schrödinger" waves or sound

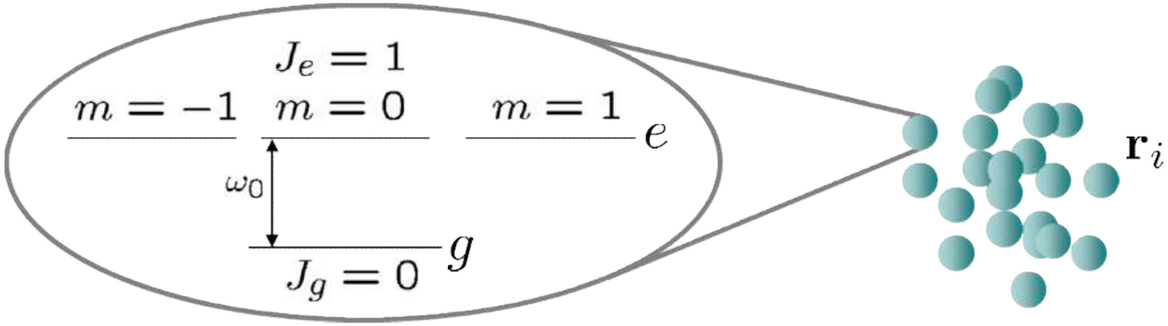


Electromagnetic waves





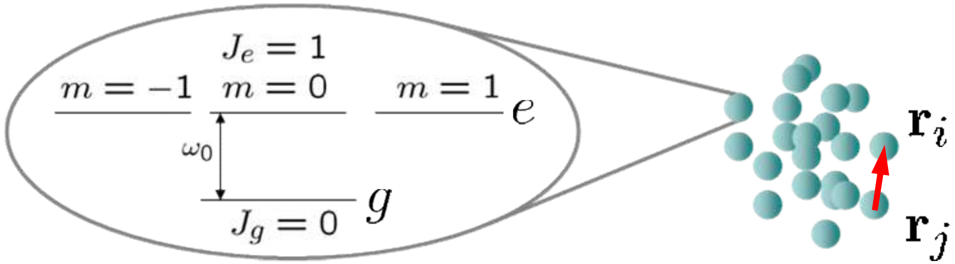
Familiar textbook Hamiltonian...



$$
\begin{aligned}
\hat{H} \;=\;& \overbrace{\sum_{i=1}^{N} \sum_{m=-1}^{1} \hbar\omega_0 |e_{im}\rangle\langle e_{im}|}^{\text{isolated atoms}} + \overbrace{\sum_{\mathbf{s}\perp\mathbf{k}} \hbar c k \left(\hat{a}^{\dagger}_{\mathbf{ks}} \hat{a}_{\mathbf{ks}} + \frac{1}{2} \right)}^{\text{free electromagnetic field}} \\
-\;& \underbrace{\sum_{i=1}^{N} \hat{\mathbf{D}}_i \cdot \hat{\mathbf{E}}(\mathbf{r}_i)}_{\substack{\text{atom-field interaction} \\ \text{in the dipole approximation}}} + \underbrace{\frac{1}{2\epsilon_0} \sum_{i\neq j}^{N} \hat{\mathbf{D}}_i \cdot \hat{\mathbf{D}}_j \delta(\mathbf{r}_i - \mathbf{r}_j)}_{\text{contact term}}
\end{aligned}
$$

Cohen-Tannoudji, Dupont-Roc & Grynberg,
Photons and Atoms: Introduction to Quantum Electrodynamics (1992)
Morice, Castin & Dalibard, PRA **51**, 3896 (1995)

Green’s matrix for light

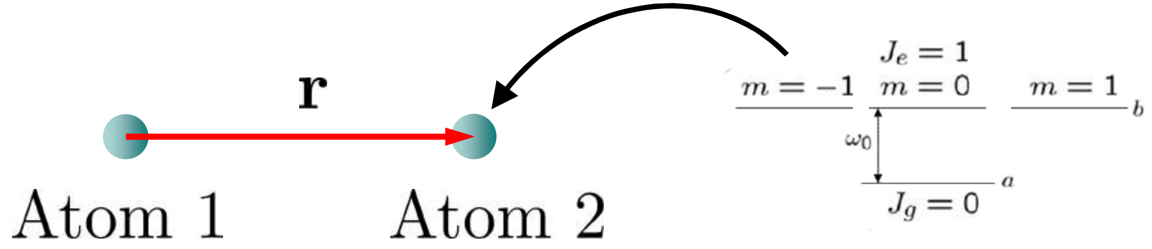


Green's matrix G describes propagation of light between pairs of atoms $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$

$$
\begin{aligned}
G_{e_{im}e_{jm'}} &= i\delta_{e_{im}e_{jm'}} - \frac{2}{\Gamma_0}(1-\delta_{e_{im}e_{jm'}}) \sum_{\mu,\nu} d^{\mu}_{e_{im}g_i} d^{\nu}_{g_j e_{jm'}} \frac{e^{ik_0 r_{ij}}}{\hbar r_{ij}^3} \\
&\times \left\{ \delta_{\mu\nu}\left[1 - ik_0 r_{ij} - (k_0 r_{ij})^2\right] - \frac{r^{\mu}_{ij} r^{\nu}_{ij}}{r_{ij}^2}\left[3 - 3ik_0 r_{ij} - (k_0 r_{ij})^2\right] \right\}
\end{aligned}
$$

$\mathbf{d}_{e_{im}g_i} = \langle J_e m | \hat{\mathbf{D}}_i | J_g 0 \rangle$ $(3N \times 3N \text{ matrix})$

Green's function in different bases

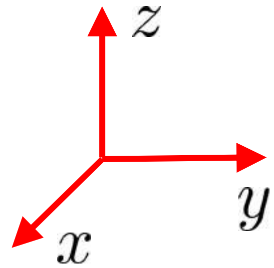


$$G_{\mu\nu}(\mathbf{r}) \quad = \quad \frac{3}{2}\frac{e^{ik_0 r}}{k_0 r}\left[P(ik_0 r)\delta_{\mu\nu} + Q(ik_0 r)\frac{r_\mu r_\nu}{r^2}\right]$$

linear basis

$\mu,\, \nu = x,\, y,\, z$

$P(x) = 1 - 1/x + 1/x^2,\ Q(x) = -1 + 3/x - 3/x^2$

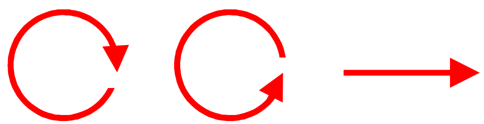


$$G_{mm'}(\mathbf{r}) \quad = \quad -\frac{4k_0^3}{3\hbar\Gamma_0}\sum_{\mu,\nu} d_{m\mu}G_{\mu\nu}(\mathbf{r})d_{m'\nu}$$

spiral basis

$m,\, m' = -1,\, 0,\, 1$

$\mathbf{d}_m = \langle J_e m|\hat{\mathbf{D}}|J_g 0\rangle$



$m = -1$ $\quad +1 \quad$ 0

Structure of the Green's matrix

$$
\begin{pmatrix}
i & 0 & 0 & G_{12}^{xx} & G_{12}^{xy} & G_{12}^{xz} & \cdots & G_{1N}^{xx} & G_{1N}^{xy} & G_{1N}^{xz} \\
0 & i & 0 & G_{12}^{yx} & G_{12}^{yy} & G_{12}^{yz} & \cdots & G_{1N}^{yx} & G_{1N}^{yy} & G_{1N}^{yz} \\
0 & 0 & i & G_{12}^{zx} & G_{12}^{zy} & G_{12}^{zz} & \cdots & G_{1N}^{zx} & G_{1N}^{zy} & G_{1N}^{zz} \\
G_{21}^{xx} & G_{21}^{xy} & G_{21}^{xz} & i & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
G_{21}^{yx} & G_{21}^{yy} & G_{21}^{yz} & 0 & i & 0 & \cdots & \cdots & \cdots & \cdots \\
G_{21}^{zx} & G_{21}^{zy} & G_{21}^{zz} & 0 & 0 & i & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
G_{N1}^{xx} & G_{N1}^{xy} & G_{N1}^{xz} & \cdots & \cdots & \cdots & \cdots & i & 0 & 0 \\
G_{N1}^{yx} & G_{N1}^{yy} & G_{N1}^{yz} & \cdots & \cdots & \cdots & \cdots & 0 & i & 0 \\
G_{N1}^{zx} & G_{N1}^{zy} & G_{N1}^{zz} & \cdots & \cdots & \cdots & \cdots & 0 & 0 & i
\end{pmatrix}
$$

One-atom dynamics:

Excitation of an isolated excited atom decays as $e^{-\Gamma_0 t}$

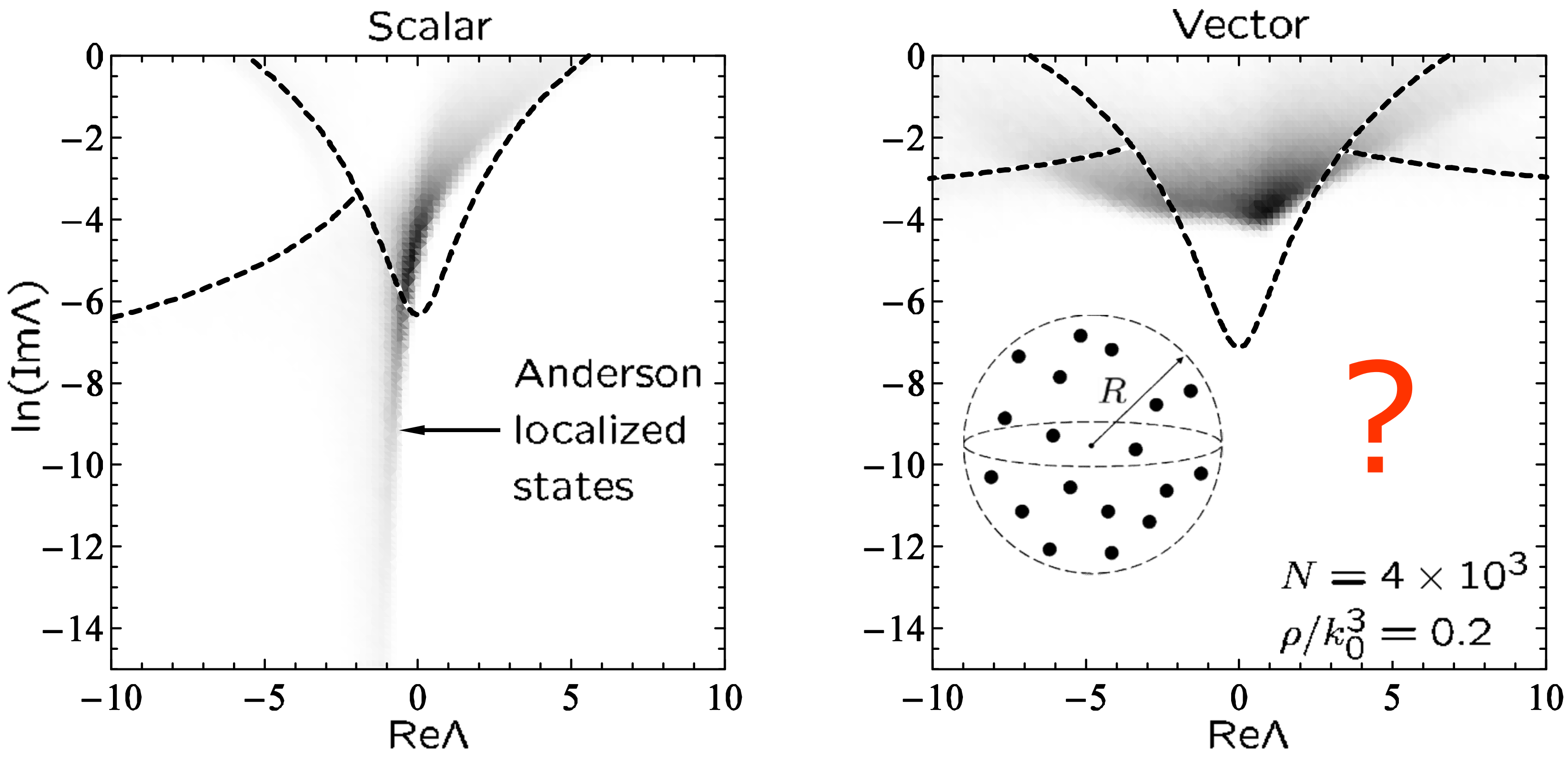
Structure of the Green's matrix

$$
\begin{pmatrix}
i & 0 & 0 & G_{12}^{xx} & G_{12}^{xy} & G_{12}^{xz} & \cdots & G_{1N}^{xx} & G_{1N}^{xy} & G_{1N}^{xz} \\
0 & i & 0 & G_{12}^{yx} & G_{12}^{yy} & G_{12}^{yz} & \cdots & G_{1N}^{yx} & G_{1N}^{yy} & G_{1N}^{yz} \\
0 & 0 & i & G_{12}^{zx} & G_{12}^{zy} & G_{12}^{zz} & \cdots & G_{1N}^{zx} & G_{1N}^{zy} & G_{1N}^{zz} \\
G_{21}^{xx} & G_{21}^{xy} & G_{21}^{xz} & i & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
G_{21}^{yx} & G_{21}^{yy} & G_{21}^{yz} & 0 & i & 0 & \cdots & \cdots & \cdots & \cdots \\
G_{21}^{zx} & G_{21}^{zy} & G_{21}^{zz} & 0 & 0 & i & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
G_{N1}^{xx} & G_{N1}^{xy} & G_{N1}^{xz} & \cdots & \cdots & \cdots & \cdots & i & 0 & 0 \\
G_{N1}^{yx} & G_{N1}^{yy} & G_{N1}^{yz} & \cdots & \cdots & \cdots & \cdots & 0 & i & 0 \\
G_{N1}^{zx} & G_{N1}^{zy} & G_{N1}^{zz} & \cdots & \cdots & \cdots & \cdots & 0 & 0 & i
\end{pmatrix}
$$

Pairwise coupling between atoms 1 & 2:

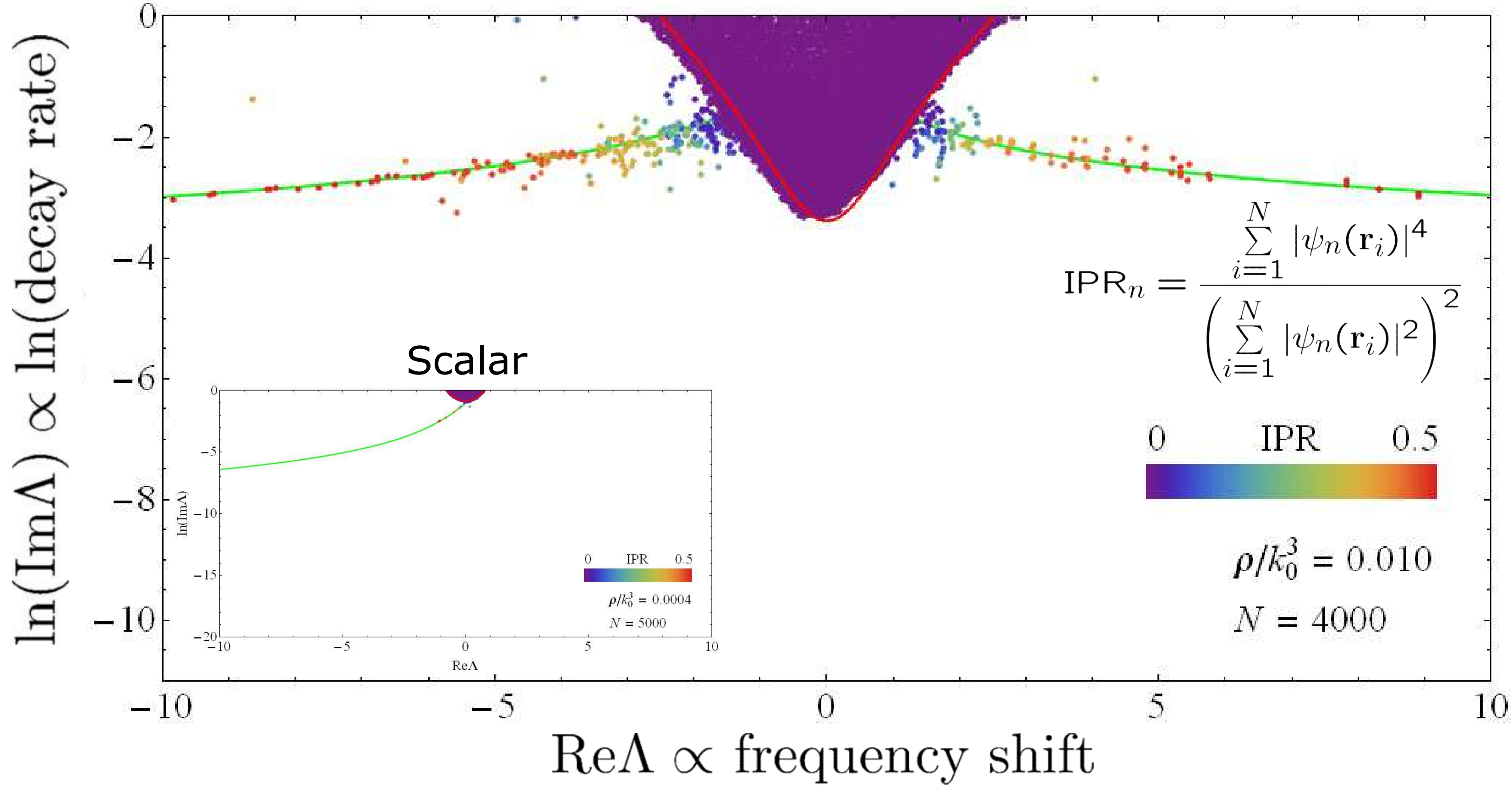
G_{12}^{xy} is the y component of the field at position 2 due to a dipole oscillating along x at position 1

Probability density of eigenvalues



Skipetrov & Sokolov, PRL **112**, 023905 (2014)

IPR for light

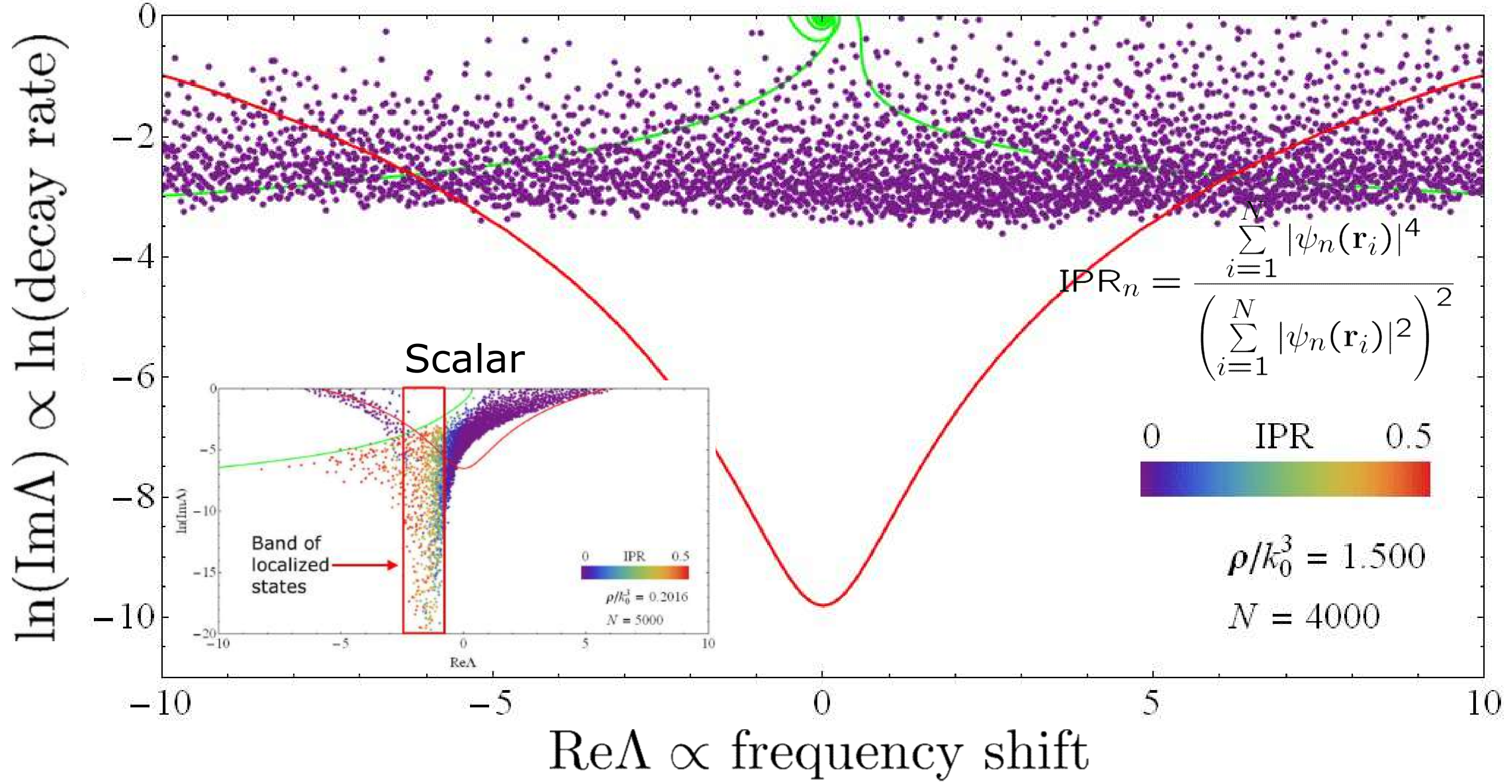


Eigenvalue domain boundary from the diffusion theory

Subradiant states localized on 2 closely located atoms

Skipetrov & Sokolov, PRL **112**, 023905 (2014)

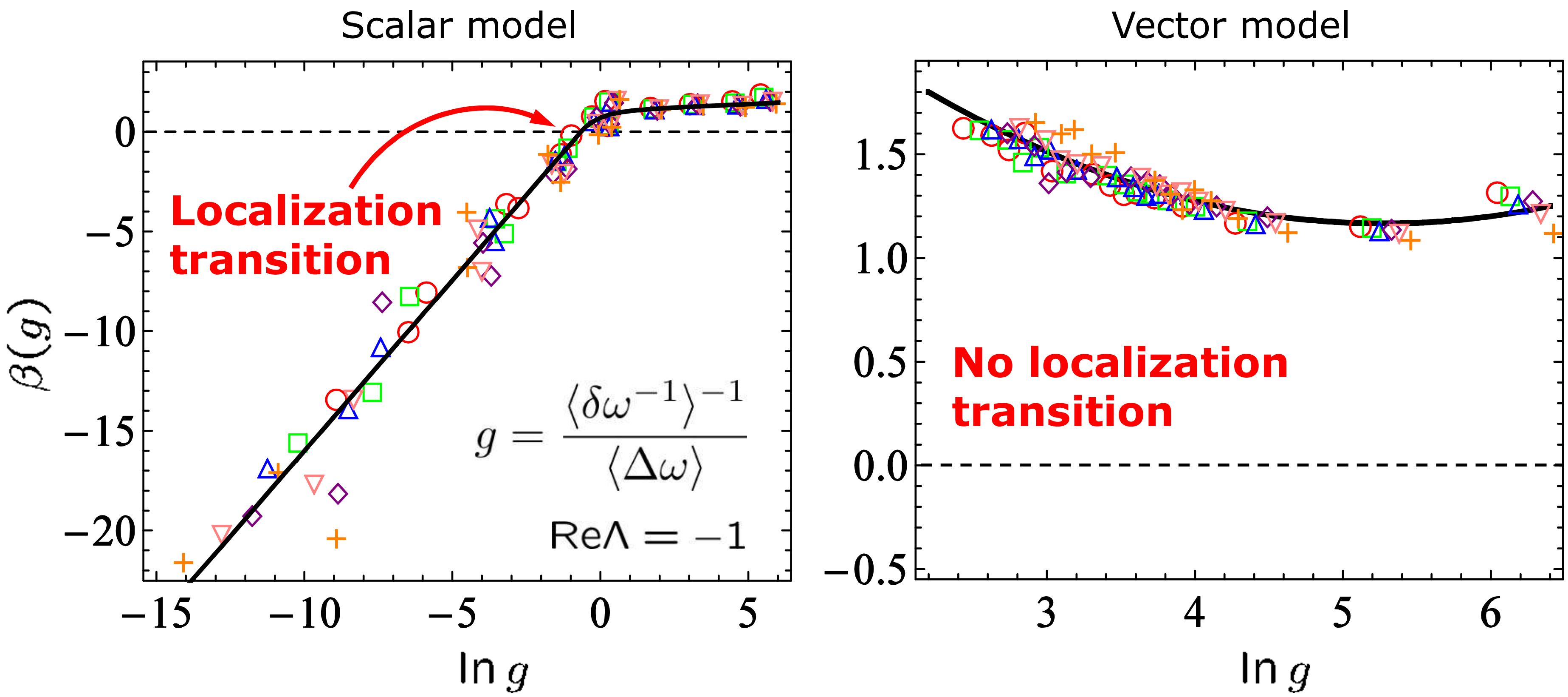
IPR for light



Eigenvalue domain boundary from the diffusion theory
Subradiant states localized on 2 closely located atoms

Skipetrov & Sokolov, PRL **112**, 023905 (2014)

No Anderson localization for light in 3D

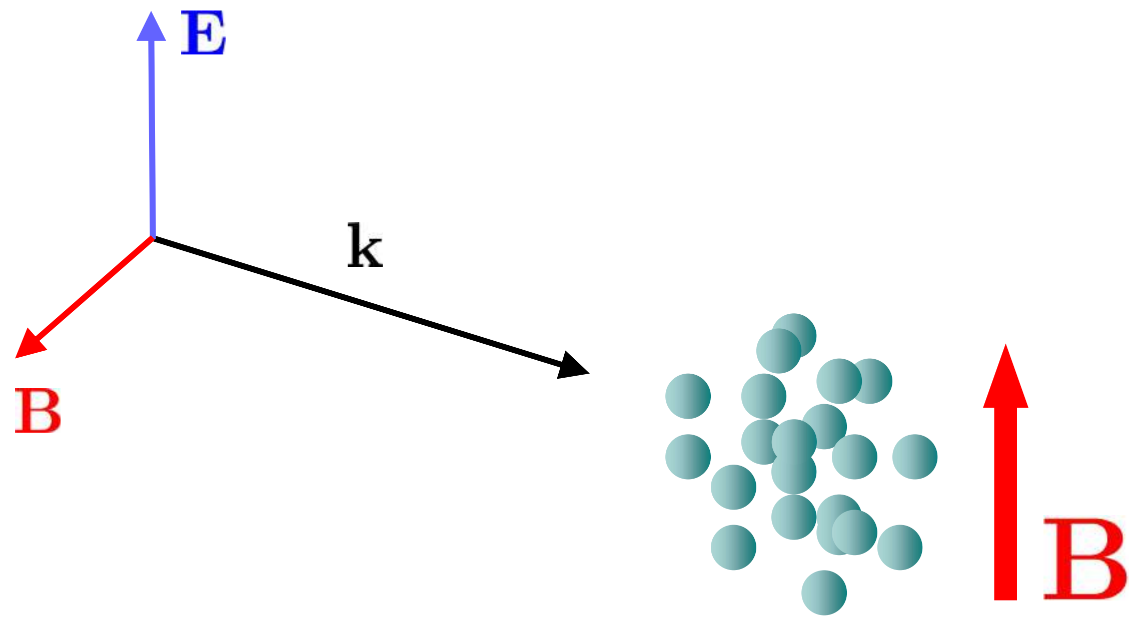


Explanation stems from near-field effects:

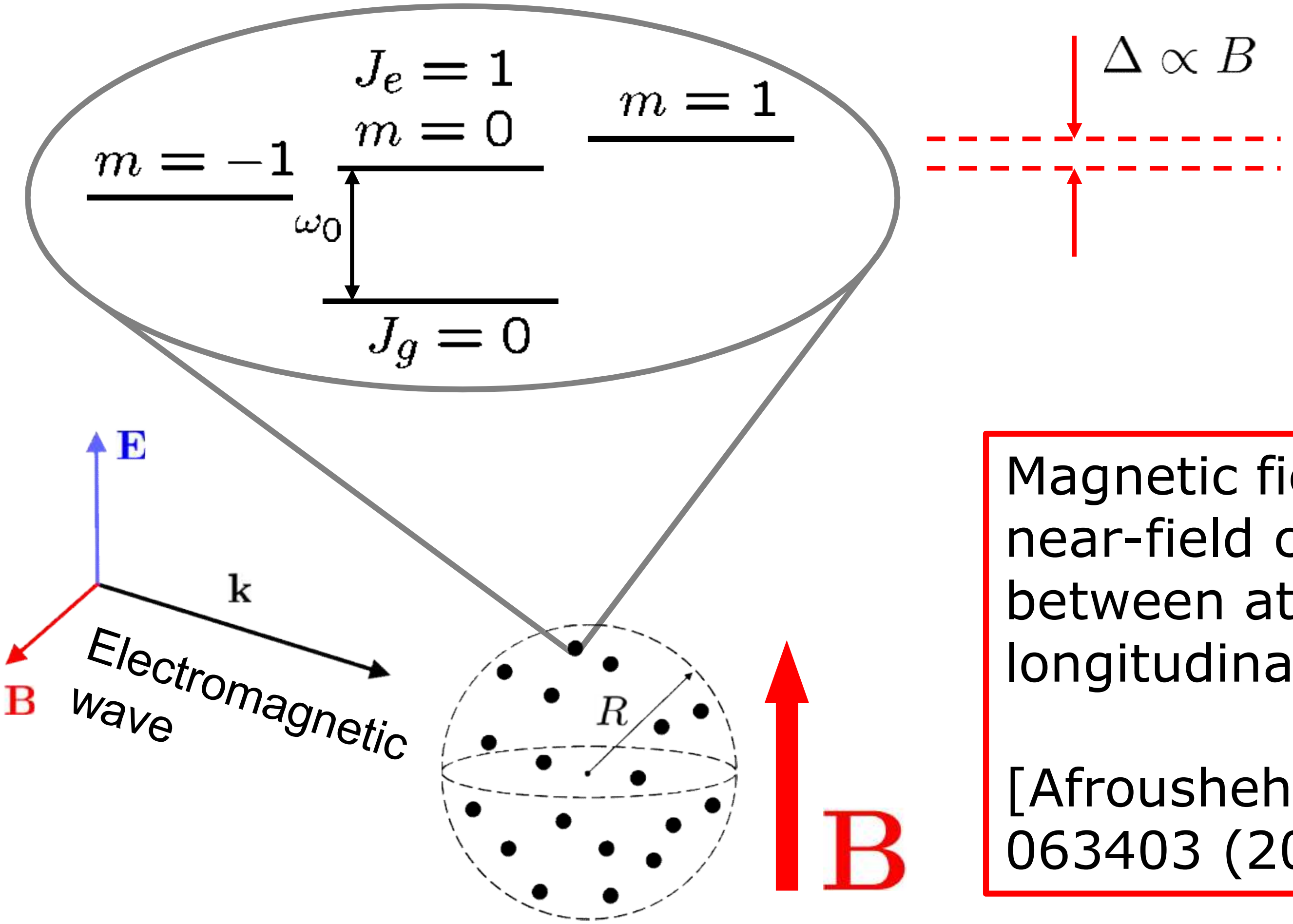
$$G_{\text{scalar}}(\mathbf{r})|_{r\to 0} \propto \frac{1}{r} \qquad \hat{G}_{\text{EM}}(\mathbf{r})\Big|_{r\to 0} \propto \frac{1}{r^3}$$

Skipetrov & Sokolov, PRL **112**, 023905 (2014)

Anderson localization of light in a magnetic field



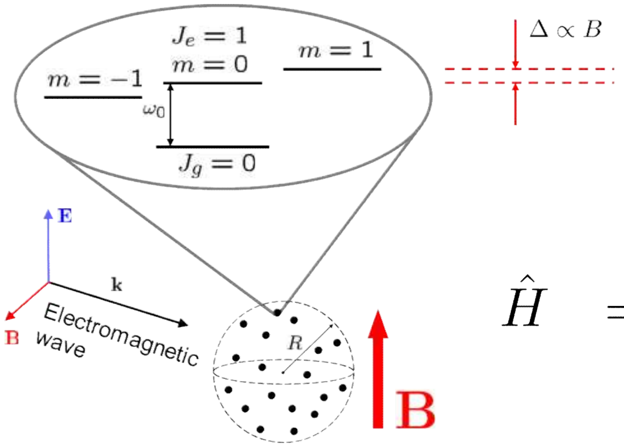
Zeemann effect



Magnetic field suppresses near-field coupling between atoms by longitudinal fields

[Afrousheh et al., PRA **73**, 063403 (2006)]

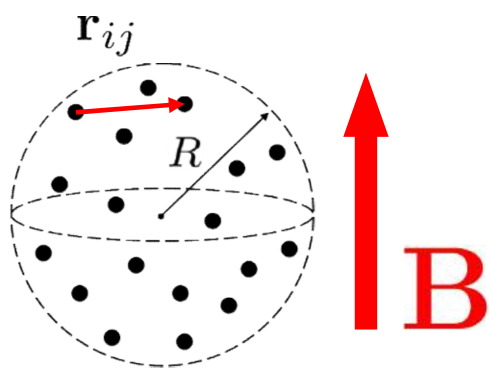
The Hamiltonian



$$
\begin{aligned}
\hat{H} \;=\; & \sum_{i=1}^{N} \sum_{m=-1}^{1} \hbar\omega_0 |e_{im}\rangle\langle e_{im}| + \sum_{\mathbf{s}\perp\mathbf{k}} \hbar ck \left(\hat{a}^{\dagger}_{\mathbf{ks}} \hat{a}_{\mathbf{ks}} + \frac{1}{2} \right) \\
& - \sum_{i=1}^{N} \hat{\mathbf{D}}_i \cdot \hat{\mathbf{E}}(\mathbf{r}_i) + \frac{1}{2\epsilon_0} \sum_{i\neq j}^{N} \hat{\mathbf{D}}_i \cdot \hat{\mathbf{D}}_j \delta(\mathbf{r}_i - \mathbf{r}_j) \\
& + \underbrace{g_e \mu_B B \sum_{i=1}^{N} \sum_{m=-1}^{1} m |e_{im}\rangle\langle e_{im}|}_{\text{Coupling of atoms with a magnetic field}}
\end{aligned}
$$

Green's matrix in a magnetic field

$$
\begin{aligned}
G_{e_{im}e_{jm'}} \quad &= \quad (i - 2m\Delta)\,\delta_{e_{im}e_{jm'}} - \frac{2}{\hbar\Gamma_0}(1 - \delta_{e_{im}e_{jm'}}) \\
&\times \quad \sum_{\mu,\nu} d^{\mu}_{e_{im}g_i} d^{\nu}_{g_j e_{jm'}} \frac{e^{ik_0 r_{ij}}}{r_{ij}^3} \\
&\times \quad \Big\{ \delta_{\mu\nu}\left[1 - ik_0 r_{ij} - (k_0 r_{ij})^2\right] \\
&- \quad \frac{r^{\mu}_{ij} r^{\nu}_{ij}}{r^2_{ij}}\left[3 - 3ik_0 r_{ij} - (k_0 r_{ij})^2\right] \Big\}
\end{aligned}
$$

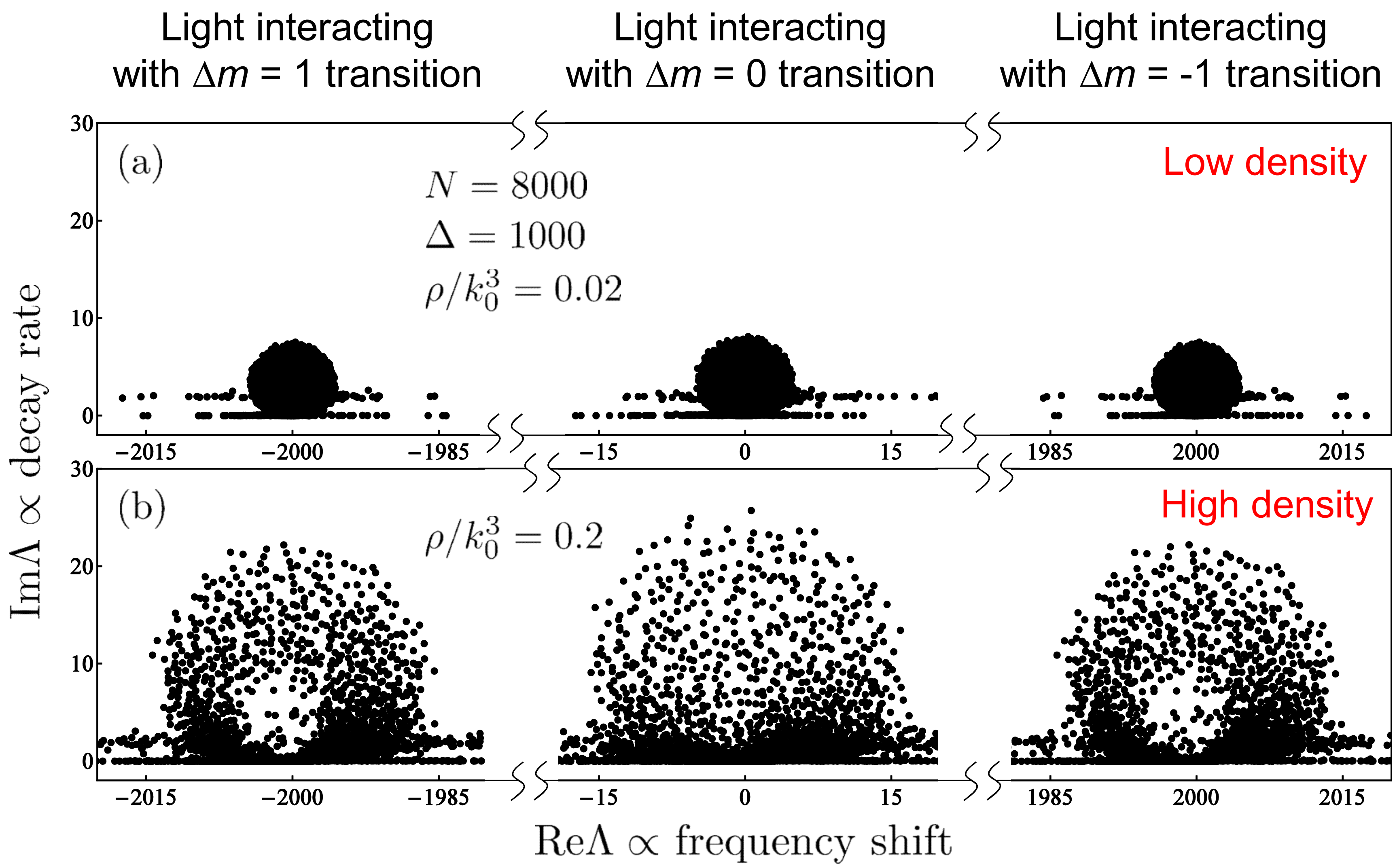


$$\Delta = g_e \mu_B B/\hbar\Gamma_0$$

$$\mathbf{d}_{e_{im}g_i} = \langle J_e m|\hat{\mathbf{D}}_i|J_g 0\rangle$$

see also Pinheiro et al., Acta. Phys. Pol. A **105**, 339 (2004)

Eigenvalues in a strong magnetic field



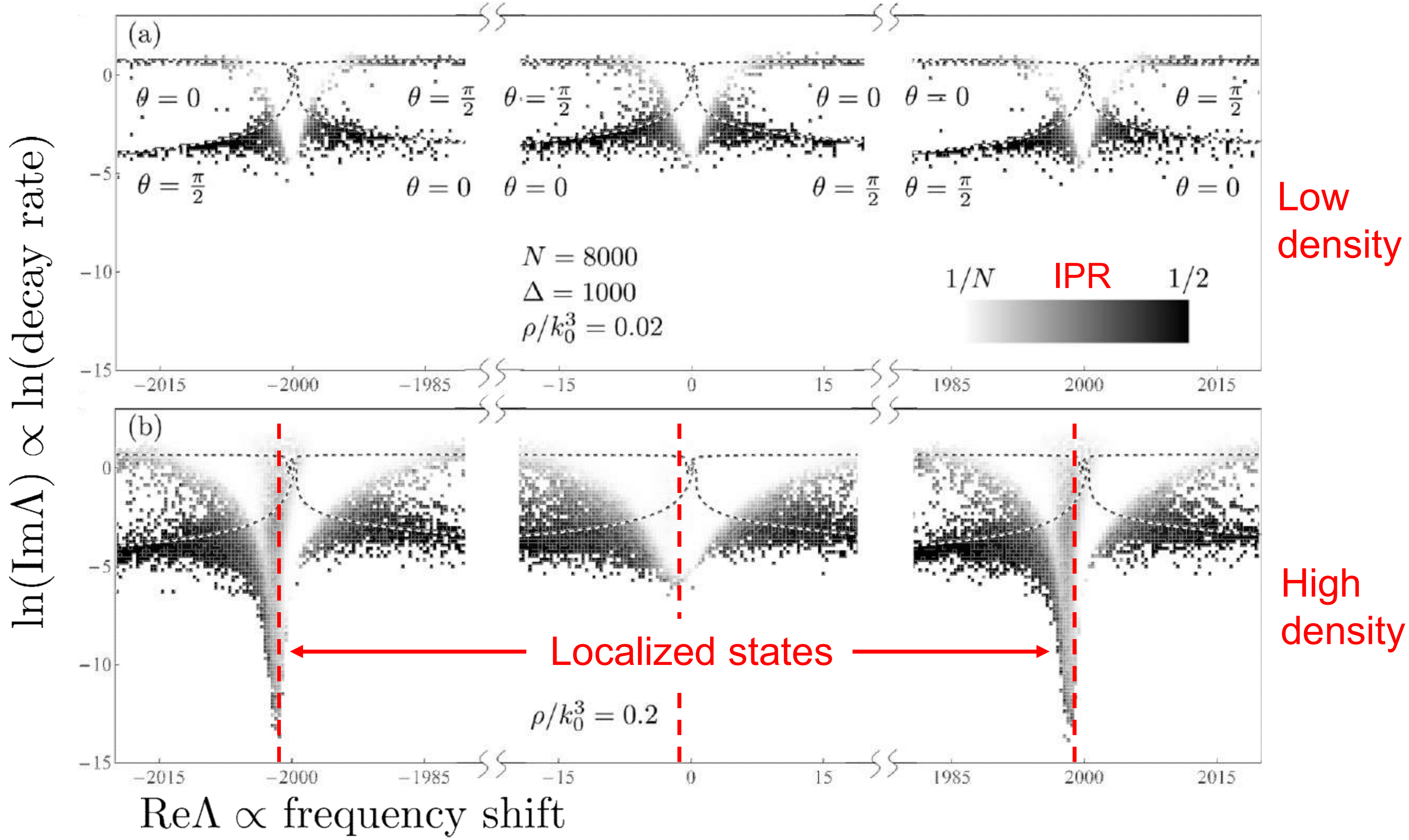
Skipetrov & Sokolov, PRL **114**, 053902 (2015)

Average inverse participation ratio

Light interacting with Δm = 1 transition

Light interacting with Δm = 0 transition

Light interacting with Δm = -1 transition



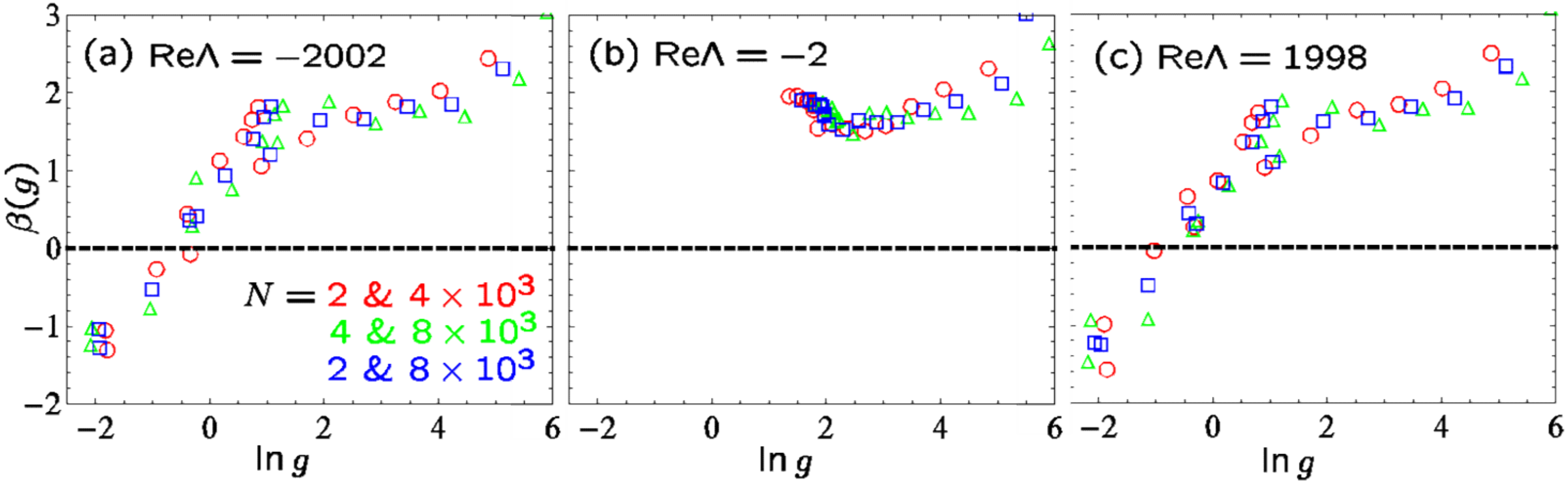
Skipetrov & Sokolov, PRL **114**, 053902 (2015)

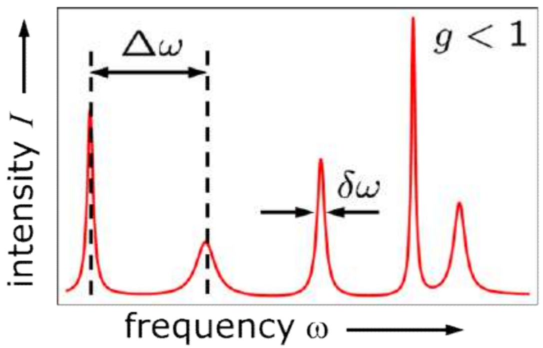
Scaling with sample size

Light interacting with Δm = 1 transition

Light interacting with Δm = 0 transition

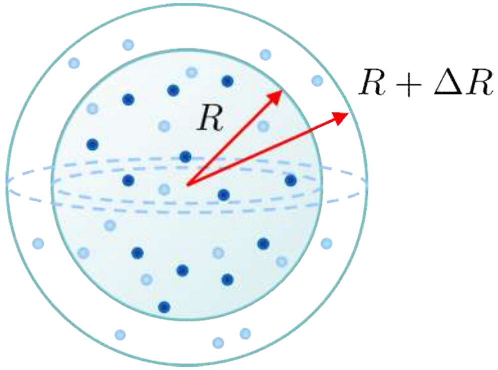
Light interacting with Δm = -1 transition





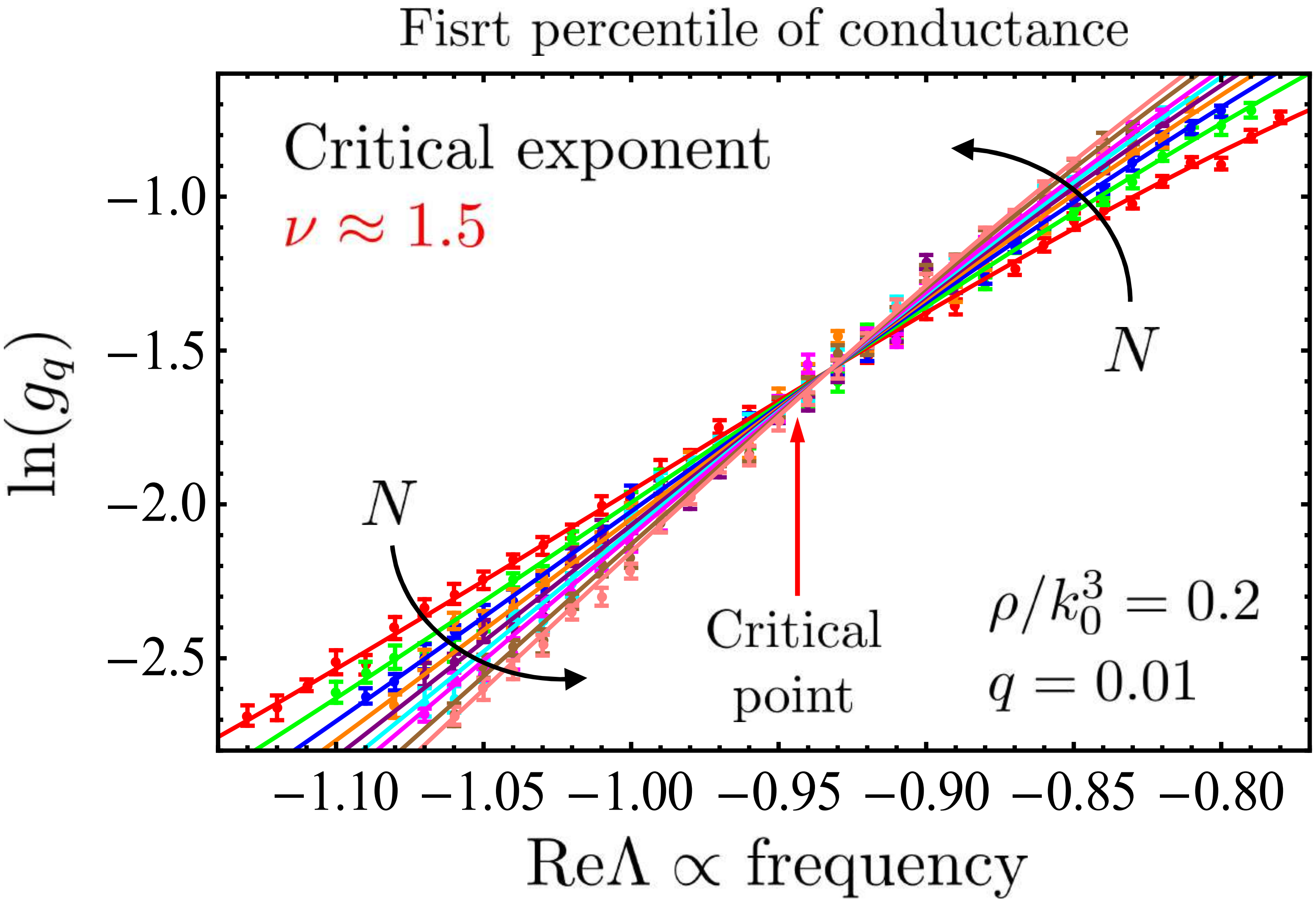
$$g = \frac{\delta\omega}{\Delta\omega}$$

$$\beta(g) = \frac{\partial \ln g}{\partial \ln k_0 R}$$



Skipetrov & Sokolov, PRL **114**, 053902 (2015)

Finite-size scaling of percentiles



Work in progress…

Anderson localization of elastic waves

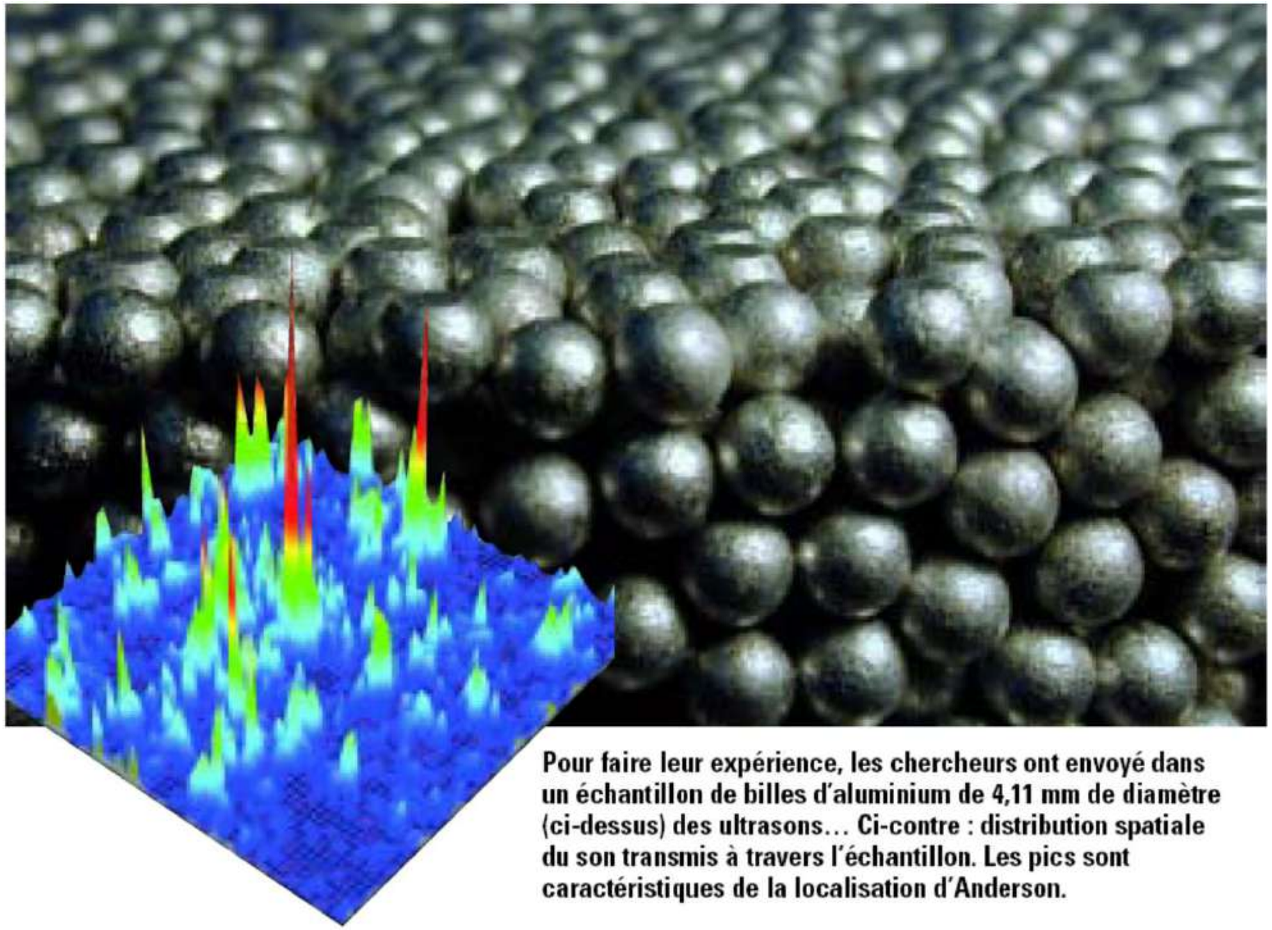


Image from *Le Journal du CNRS* (December 2008)

Elastic wave equation

$$\rho\omega^2 u_i + \frac{\partial}{\partial x_j}\left(c_{ijkl}\frac{\partial}{\partial x_k}u_l\right) = -f_i$$

mechanical
displacement
in the direction *i*

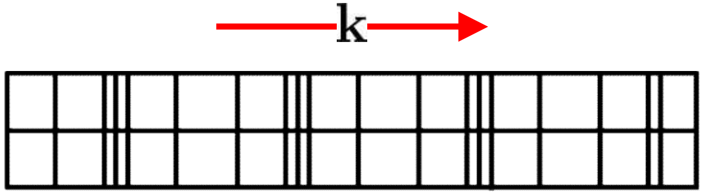
elasticity
tensor

excitation

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

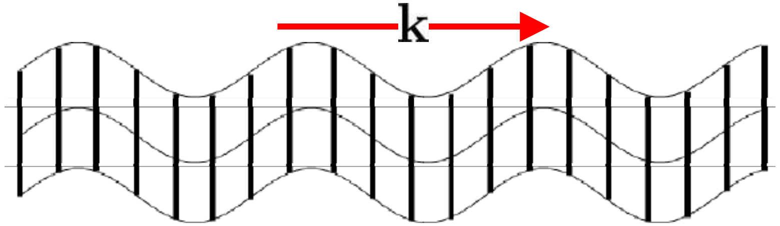
Lamé parameters

compressional waves:



velocity $\alpha = \sqrt{\frac{\lambda+2\mu}{\rho}}$

shear waves:



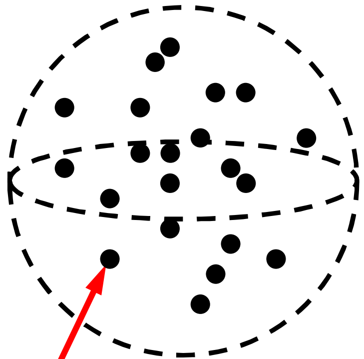
velocity $\beta = \sqrt{\frac{\mu}{\rho}}$

Point-scatterer model

Real sample

Hu et al., Nature Physics **4**, 945 (2008)

Model



Identical aluminum beads
with many resonances

Identical point scatterers
with a single resonance

Elastic Green's function

$$\widehat{G}(\mathbf{r}) = \frac{k_\alpha}{4\pi(\lambda+2\mu)} \times \left\{ \frac{e^{ik_\alpha r}}{3k_\alpha r} [\mathbb{1} + (\mathbb{1} - 3\widehat{r}\otimes\widehat{r})] \left(-1 - \frac{3i}{k_\alpha r} + \frac{3}{(k_\alpha r)^2}\right) - \left(\frac{\alpha}{\beta}\right)^3 \frac{e^{ik_\beta r}}{3k_\beta r} [-2\mathbb{1} + (\mathbb{1} - 3\widehat{r}\otimes\widehat{r})] \left(-1 - \frac{3i}{k_\beta r} + \frac{3}{(k_\beta r)^2}\right) \right\}$$

$k_\alpha = \frac{\omega}{\alpha}$, $k_\beta = \frac{\omega}{\beta}$, $\widehat{r} = \frac{\mathbf{r}}{r}$

Typically, $\alpha > \beta$ ($\alpha/\beta \simeq 2$ for aluminium)

Equipartition principle:

$$\frac{\langle \text{Energy of shear waves} \rangle}{\langle \text{Enerfy of compressional waves} \rangle} = 2\left(\frac{\alpha}{\beta}\right)^3 > 1$$

Elastic Green's function in the near field

$$\hat{G}(\mathbf{r}) = \frac{k_\alpha}{4\pi(\lambda+2\mu)} \times \left\{ \frac{e^{ik_\alpha r}}{3k_\alpha r} [\mathbb{1} + (\mathbb{1} - 3\hat{r}\otimes\hat{r})] \left(-1 - \frac{3i}{k_\alpha r} + \frac{3}{(k_\alpha r)^2}\right) - \left(\frac{\alpha}{\beta}\right)^3 \frac{e^{ik_\beta r}}{3k_\beta r} [-2\mathbb{1} + (\mathbb{1} - 3\hat{r}\otimes\hat{r})] \left(-1 - \frac{3i}{k_\beta r} + \frac{3}{(k_\beta r)^2}\right) \right\}$$

$$\Downarrow$$

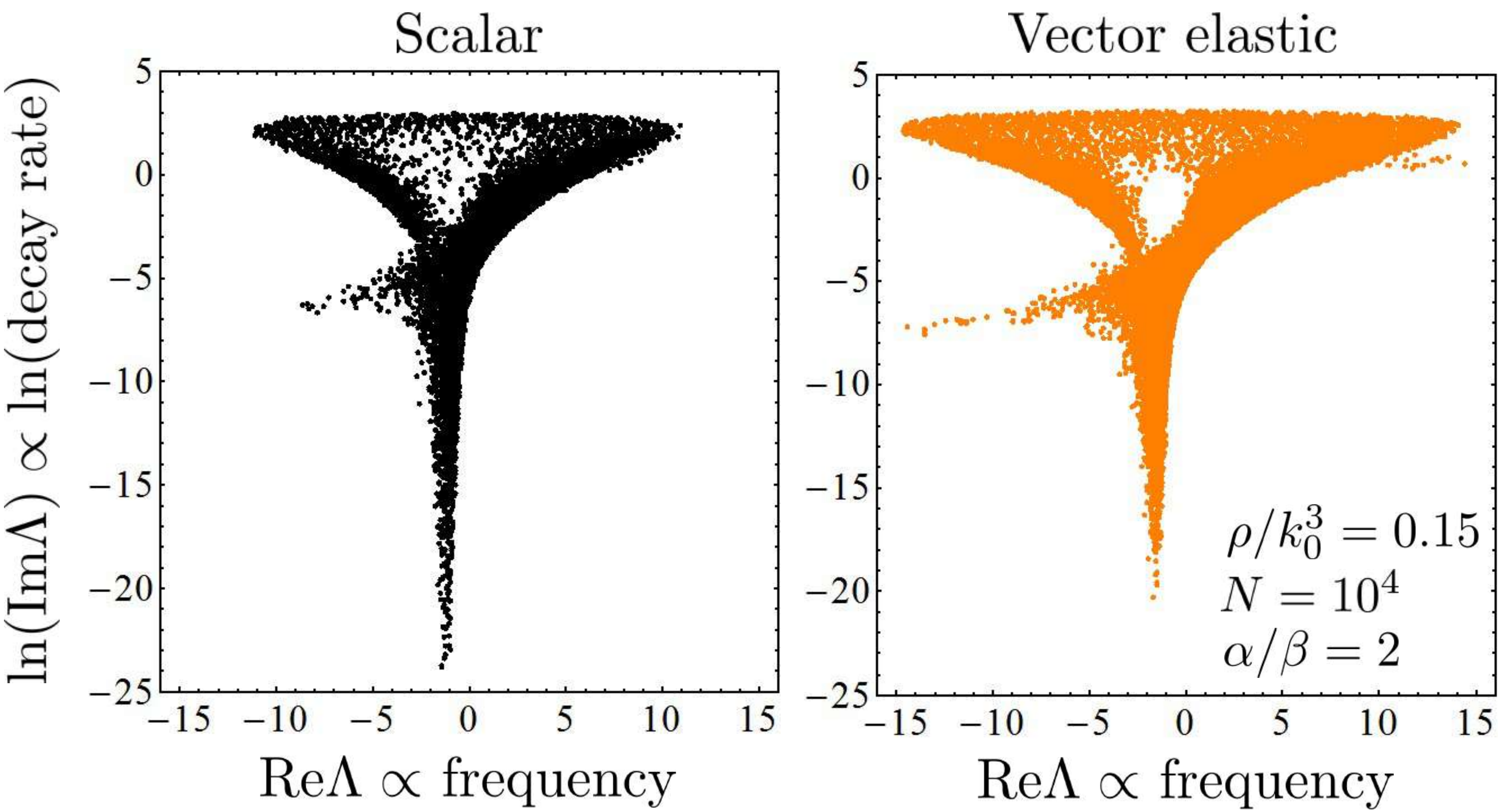
Near-field behavior:

$$\hat{G}(\mathbf{r})\Big|_{r\to 0} = -\frac{1}{8\pi\mu}\left(1 - \frac{\beta^2}{\alpha^2}\right)\frac{\mathbb{1} - 3\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}}{r} \propto \frac{1}{r}$$

Similar to the scalar case and different from the electromagnetic one:

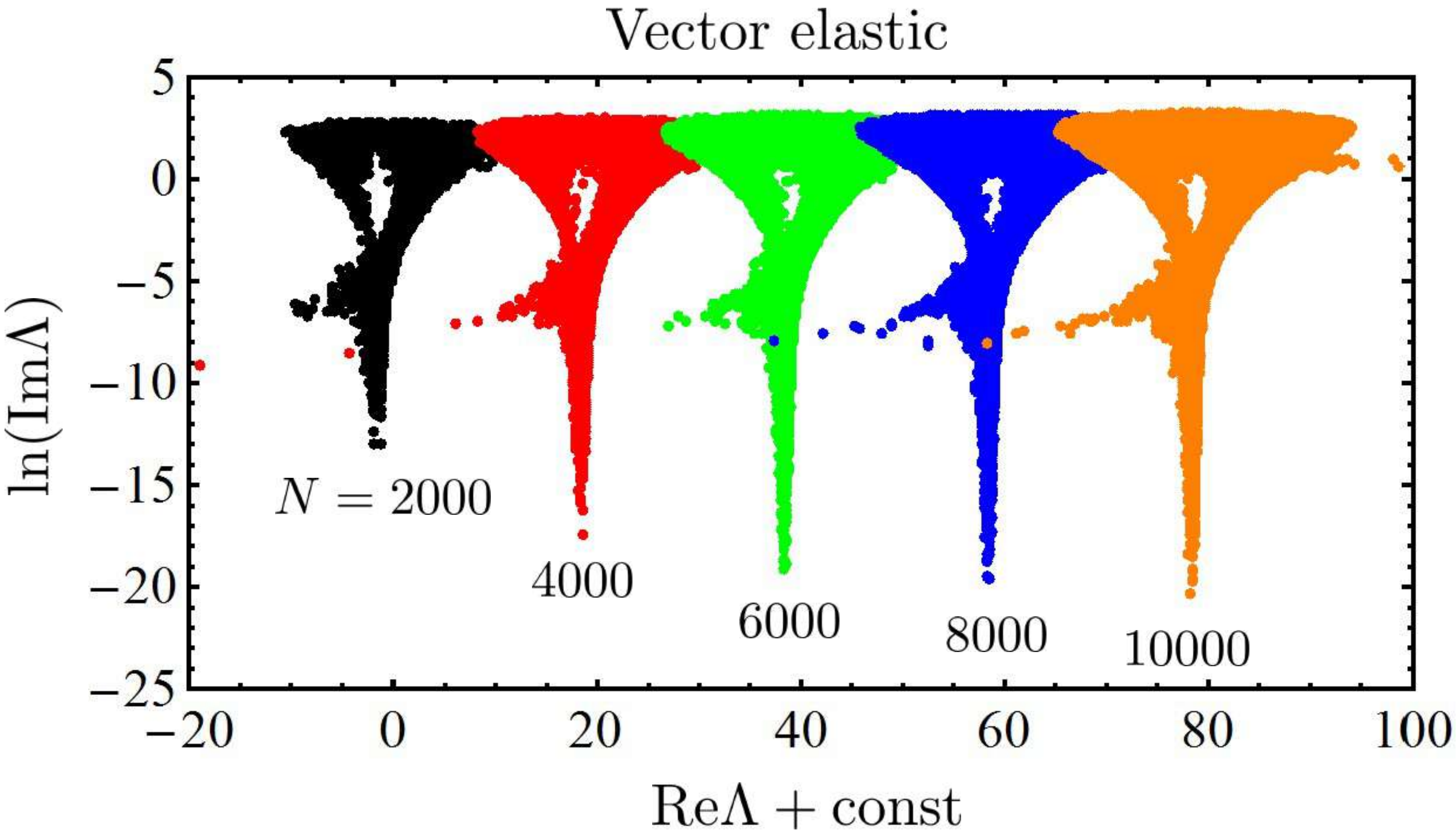
$$\hat{G}_{\mathrm{EM}}(\mathbf{r})\Big|_{r\to 0} \propto \frac{1}{r^3}$$

Eigenvalues: scalar vs elastic



Work in progress…

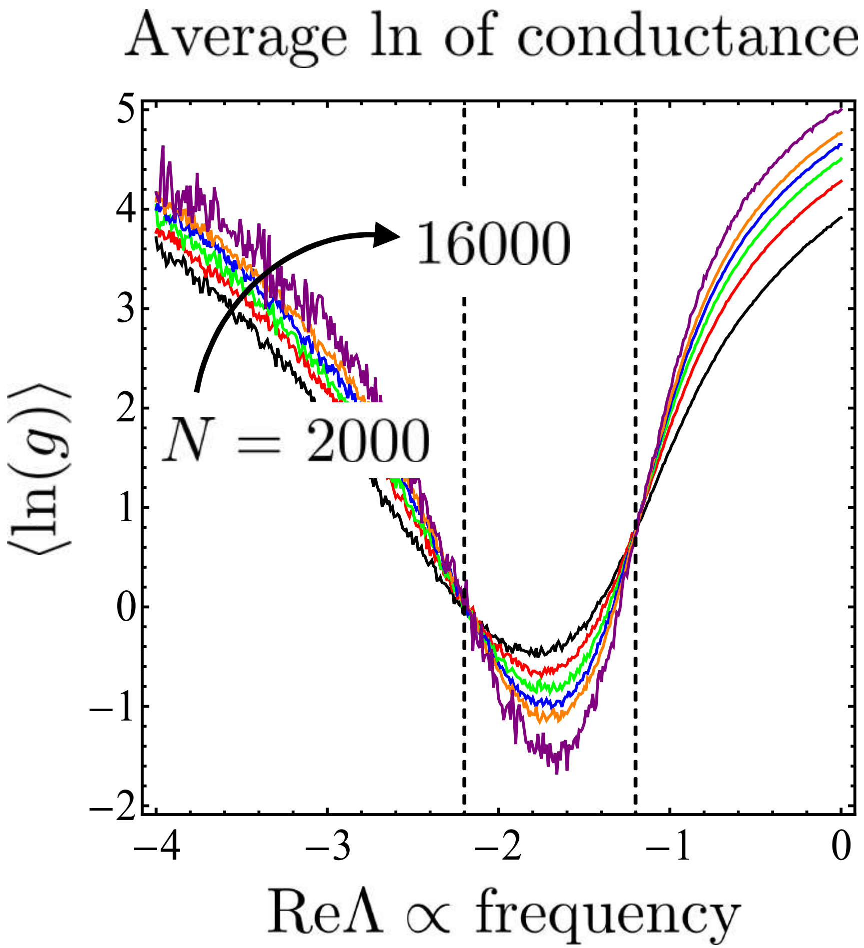
Eigenvalues: evolution with sample size

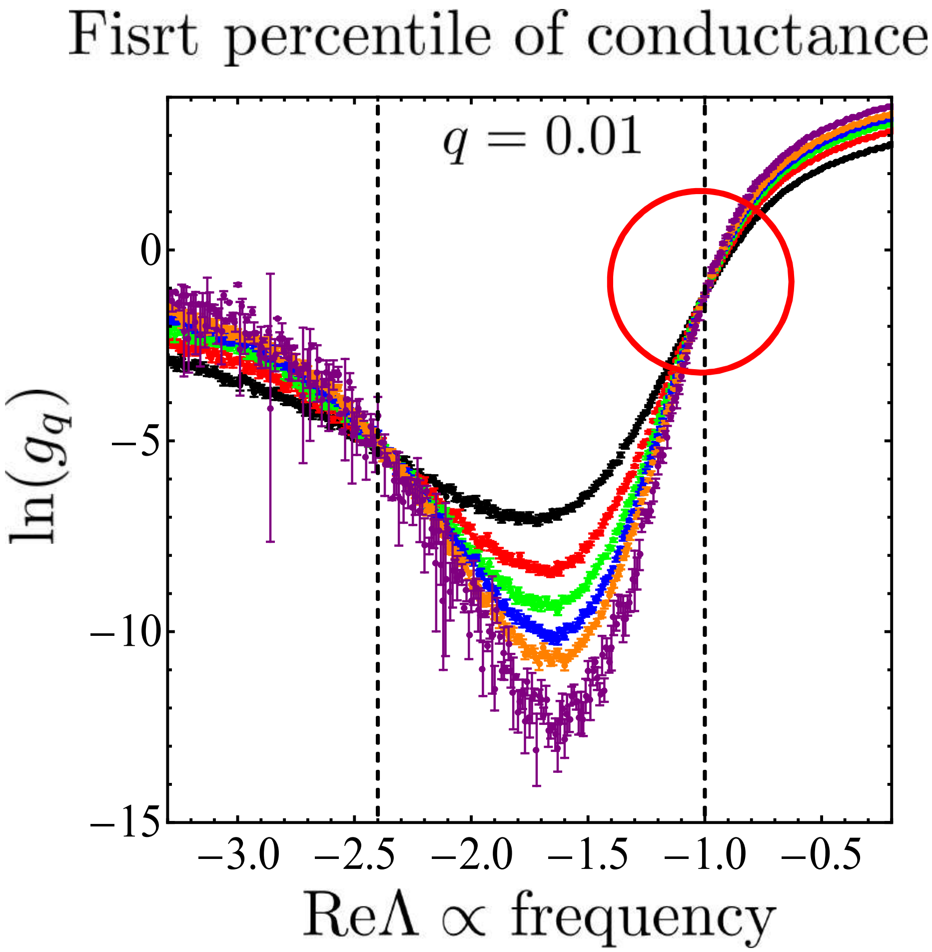


$\rho/k_0^3 = 0.15$
$\alpha/\beta = 2$

Work in progress…

Scaling for elastic waves



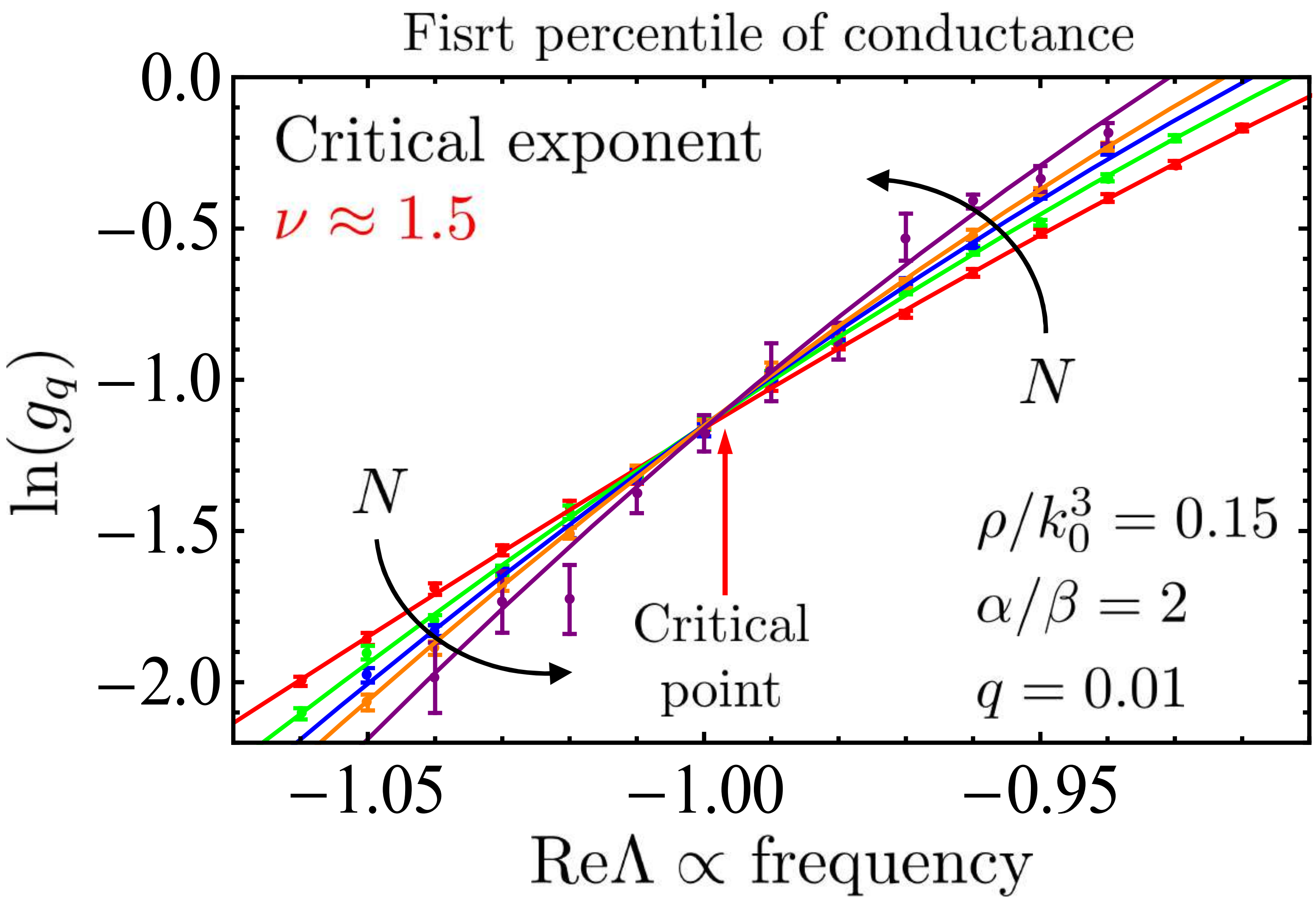


$\rho/k_0^3 = 0.15$

$\alpha/\beta = 2$

Work in progress...

Finite-size scaling of percentiles



Work in progress…

Conclusions

- Wave scattering by random ensembles of resonant point scatterers is naturally described by **Euclidean random matrix (ERM) models**
- ERM models can be used to study **Anderson localization transitions** with account for peculiarities specific for a particular type of waves: scalar vs vector waves, light vs elastic waves, etc.
- **Anderson localization of light by atoms** is possible only in the presence of a strong magnetic field and that the localization transition for elastic waves is similar to that of scalar waves
- More developments of **analytic approaches to ERM** are necessary for further progress

Thank you for your attention!